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SOLID ANALYTICAL GEOMETRY AND DETERMINANTS

BY THE SAME AUTHOR

Plane Trigonometry

This book emphasizes the importance of the function concept for elementary trigonometry. Cloth; 6 by 9 inches; 110 pages; 58 figures.

PUBLISHED BY

JOHN WILEY & SONS, INC.
NEW YORK

SOLID ANALYTICAL GEOMETRY AND DETERMINANTS

BY

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NEW YORK
JOHN WILEY & SONS, Inc.
London: CHAPMAN & HALL, Limited
1930

Printed in U. S. A.

Printing
F. H. GILSON CO.
BOSTON

Composition and Plates
TECHNICAL COMPOSITION CO.
CAMBRIDGE

Binding
STANHOPE BINDERY
BOSTON

PREFACE

A three-hour course in Solid Analytical Geometry is offered for students in the junior year in many of the colleges and universities in this country. Though books on Plane Analytical Geometry frequently devote some chapters to the geometry of a space of three dimensions, the material covered in these chapters is, with few exceptions, not intended to do more than provide a general introduction to the subject, so as to enable students to understand the references to it which have to be made in courses on the calculus. But it rarely goes far enough to acquaint them with the more interesting and valuable methods of this field.

For many years, while teaching this subject at the University of Wisconsin and at Swarthmore College, it has seemed to the author that, in the study of Solid Analytical Geometry, the young student of mathematics can find an excellent opportunity for an introduction to methods and principles which have an important part in various fields of advanced mathematics. Among these are the methods based on the theory of determinants and on the concept of the rank of a matrix. In more advanced mathematical subjects these theories are developed and used with a great measure of generality; they find relatively simple application in the subject to which this book is devoted. Unfortunately, there are not readily accessible for use in undergraduate classes treatments of these theories which are on the one hand adequate for the uses to be made of them here and on the other hand not too advanced to be available for an introductory purpose.

For these reasons, the first chapter of this book presents an exposition of some of the properties of determinants and matrices, followed in Chapter II by a treatment of systems of linear equations. The latter subject is not carried so far as to include the most general case, but, it is hoped, far enough to serve in the later chapters. The repeated applications of the results of the first two chapters which are made in the subsequent work (as evidenced by the numerous references to Chapters I and II) should indicate their importance. With the basis thus provided it becomes pos-

sible to deal with the geometrical questions of the later chapters in a way which lends itself readily to extension to problems of a more general character. Thus the discussion of the theory of quadric surfaces in Chapters VII and VIII may be made to serve as an introduction to the theory of quadratic forms in n variables. Having studied these chapters, the reader should be able to proceed to the well-known excellent books, by Bôcher and by Dickson, mentioned in the introductory paragraph of Chapter I. To these books the author owes a large debt. The spirit which pervades them has been a guide for him; and it would be a source of gratification if the present book were to lead its readers to more extended study of the subjects treated by these authors.

Chapters III to X deal with the loci of equations of the first and second degree in three variables from the point of view of real, metric geometry. Elements at infinity and complex elements are considered as non-existent. This point of view has been taken because, in the author's judgment, a satisfactory treatment of the questions which arise through the inclusion of such elements can only be made after the explicit adoption of adequate bases on which projective geometry and complex geometry can be erected. Since this would involve quite a different orientation than the scope of the present book permits, it was deemed better to proceed on the implicit assumptions of real metric geometry on which the student's earlier work in geometry may be supposed to have been founded.

It is only from this point of view that the detailed classification of quadric surfaces made in Chapter VIII can be justified; and only when it is kept in mind, can statements like the one in Theorem 12 on page 175, to the effect that there are no lines on a non-singular quadric with negative discriminant, be explained. It must however be recognized that there occur instances in which, if only for the sake of emphasis, infinite elements and complex elements must be mentioned.

In subject matter the last eight chapters follow largely the traditional content of introductory courses in Solid Analytical Geometry. The treatment introduces modes of procedure and devices which have been developed in the course of the many years during which the author has taught the subject and which have probably also been used by other teachers.

It will usually be most satisfactory to work through the greater part of Chapters I and II before Chapter III is started. But it may be found desirable, as has frequently been the author's practice, to begin by spending one hour a week on the geometric part of the book, beginning with Chapter III, while the remainder of the time is given to the algebraic work of the first two chapters.

The exercises form an integral part of the course which this book presents. The author has not hesitated therefore to refer, in a number of cases (see e.g., pages 116, 222, and 228), to results established in an exercise. A good many of the problems serve no other purpose than that of illustrating the material in the text. But there are other problems, and these are doubtless the more valuable ones, which require a certain amount of original thinking.

Thanks are due to other authors besides those which were mentioned in a preceding paragraph; but the uses which I have made of their work are too indefinite in character to make explicit references possible. It is a pleasure to acknowledge my indebtedness to the mathematical library of Brown University for allowing photographs to be made of the models in its possession for the illustrations of quadric surfaces which appear in this book. Furthermore I wish to express my thanks to Mr. George B. Hoadley, a senior student in Swarthmore College, who has drawn the figures, and to Miss Alice M. Rogers, research assistant at the Sproul Observatory of Swarthmore College, who has given valuable aid in the reading of proofs. This preface would not be complete without a word of appreciation for the unfailing courtesy and patience which the publishers and the printers have contributed to the production of this book.

ARNOLD DRESDEN.

SWARTHMORE COLLEGE,
February, 1930.

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SOLID ANALYTICAL GEOMETRY AND DETERMINANTS

CHAPTER I

DETERMINANTS AND MATRICES

1. Introduction. The Study of Solid Analytical Geometry, to which this book is chiefly devoted, leads repeatedly to the problem of solving systems of linear equations in several variables, in which the number of variables may be less than, equal to, or greater than the number of equations. The methods for dealing with this problem which are found in books on elementary algebra and in college algebra texts are not sufficiently general in character to suit the needs of our subject. A more complete treatment of the theory of determinants than is found in such books becomes necessary. For this reason, and also on account of the manifold uses of determinants in various fields of mathematics, finally because of the great intrinsic interest of the subject, the first chapter of this book will be devoted to an introduction to the theory of determinants and to a few ideas concerning matrices. This will be followed in Chapter II by a treatment of systems of linear equations. In this treatment the problem is not considered in its complete generality, but in a form sufficiently inclusive to suit the needs of the later chapters in this book. The reader who desires to pursue this subject further can do so in two excellent books, dealing with advanced topics in algebra, viz., Bôcher, Introduction to Higher Algebra, and Dickson, Modern Algebraic Theories.

2. Definitions and Notations.

DEFINITION I. **A determinant is a square array of numbers to which a single number, called the *value of the determinant*, is attached by the method stated in Definition V.**

Vertical bars are placed on either side of the array. The symbol so obtained is used also to designate the number that is to be asso-

ciated with the array. For example, the symbols

$$\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 4 & -1 & 2 \\ 0 & 3 & -5 \\ -2 & 1 & 4 \end{vmatrix}$$

with each of which a single number is associated in accordance with Definition V are determinants. The same symbols are used to designate the values of these determinants.

Definition II. **The numbers in the square array which constitutes the determinant are called its *elements*; the horizontal lines in the array are called *rows*, the vertical lines *columns*; the diagonal which runs from upper left to lower right is called the *principal diagonal*, the other diagonal is called the *secondary diagonal*.**

Definition III. **The *order* of a determinant is the number of elements in any one row or column.**

Remark. A determinant of order n is made up of n^2 elements.

Notations. In the general form of a determinant every element has affixed to it two indices; the first of these designates the row in which the element stands and is called the **row index,** the second designates the column of the element and is called the **column index.** The general forms of the determinants of the third and fourth order will therefore be as follows:

$$(1)\quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \qquad (2)\quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

A determinant of the nth order, where n designates a positive integer, in the most general form will appear as follows:

$$(3)\quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

These are rather lengthy symbols; they make it desirable to have more compact symbols which can be used when it is not necessary to designate the elements of the determinant explicitly. In such cases we frequently designate a determinant by merely writing the elements of the principal diagonal. Thus the symbols $|a_{11}a_{22}a_{33}|$, $|a_{11}a_{22}a_{33}a_{44}|$ and $|a_{11}a_{22}\ \ldots\ a_{nn}|$ are used

to designate the determinants (1), (2), and (3) respectively. A still shorter way of representing the general determinant consists in writing a single element with literal indices and indicating the values which these indices are to take. In this notation the determinants (1), (2), and (3) would be represented by the notations $|a_{ij}|$, $i, j = 1, 2, 3$; $|a_{ij}|$, $i, j = 1, 2, 3, 4$; and $|a_{ij}|$, $i, j = 1, 2, \ldots, n$ respectively.

3. The Value of a Determinant.

Definition IV. **Whenever in a set of numbers, consisting of integers from 1 upward in arbitrary order, a larger integer precedes a smaller one, we say that there is an *inversion*.**

For example, in the row of indices

$$4 \quad 6 \quad 3 \quad 2 \quad 5 \quad 1$$

there are 11 inversions: 3 inversions because the number 4 is followed by the smaller numbers 3, 2 and 1; 4 inversions because 6 is followed by 3, 2, 5 and 1; 2 inversions because 3 precedes 2 and 1; 1 inversion because 2 precedes 1, and 1 inversion because 5 precedes 1.

Definition V. **The *value of a determinant* is the algebraic sum of all possible products obtainable by taking one and only one factor from each row and from each column, preceded by the plus or minus signs, according as the number of inversions of the column indices of the factors of a product are even or odd, when the row indices are in the natural order 1, 2, 3, etc.**

The indicated sum of these products is called the ***expansion of the determinant.***

Remark 1. We must remember that, although the same symbol is used for a determinant and for the value of this determinant, the concepts "determinant" and "value of a determinant" are distinct concepts; the latter is a number; the former is a square array of numbers with which a number is associated according to Definition V.

Remark 2. The expansion of the general determinant of the nth order (3) will therefore consist of terms of the form $a_{1c_1}a_{2c_2} \ldots a_{nc_n}$, in which $c_1\, c_2 \ldots c_n$ is some permutation of the set of numbers $1, 2, \ldots n$; this term will be preceded by the plus or minus sign, according as the number of inversions of the set $c_1\, c_2 \ldots c_n$ is even or odd. Since the number of permutations of the set of

integers 1, 2, . . . n is $n!$, it follows that the expansion of the general determinant (3) consists of $n!$ terms.

Examples.

1. The value of the second order determinant $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is the algebraic sum of two terms; each term must contain two factors, one and only one from each row and from each column. If we take a_{11} from the first row, we must take a_{22} from the second; thus we get the product $a_{11}a_{22}$. If we take a_{12} from the first row, we must take a_{21} from the second, so that we obtain the product $a_{12}a_{21}$. In both these products the row indices are in the natural order 1,2. In the first product the set of column indices is 1,2, which has no inversions; the column indices in the second product form the set 2,1, which has 1 inversion. Consequently the product $a_{11}a_{22}$ is preceded by the plus sign, and the product $a_{12}a_{21}$ is preceded by the minus sign. Therefore

$$| a_{11} \quad a_{22} | = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

2. The value of the third order determinant (1) is obtained as the algebraic sum of 6 products; if we write the factors of each product in the order of their row indices these products are $a_{11}a_{22}a_{33}$, $a_{11}a_{23}a_{32}$, $a_{12}a_{21}a_{33}$, $a_{12}a_{23}a_{31}$, $a_{13}a_{21}a_{32}$, $a_{13}a_{22}a_{31}$. The numbers of inversions in the column indices of these products are 0,1,1,2,2 and 3 respectively. We conclude that

$$| a_{11}a_{22}a_{33} | = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

3. To determine the value of the determinant $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$ we write down every possible product of four factors, in each of which there is one and only one element from each row and from each column. We order the factors in each product according to the rows from which they are taken, and indicate below them the columns to which they belong. The number of inversions in the column indices then determines the sign to be prefixed to each product, in accordance with the rule laid down in Definition V. Thus we obtain the following expansion:

$$\begin{aligned}
&+ \underset{1}{1}\cdot\underset{2}{2}\cdot\underset{3}{6}\cdot\underset{4}{20} - \underset{1}{1}\cdot\underset{2}{2}\cdot\underset{4}{10}\cdot\underset{3}{10} - \underset{1}{1}\cdot\underset{3}{3}\cdot\underset{2}{3}\cdot\underset{4}{20} + \underset{1}{1}\cdot\underset{3}{3}\cdot\underset{4}{10}\cdot\underset{2}{4} + \underset{1}{1}\cdot\underset{4}{4}\cdot\underset{2}{3}\cdot\underset{3}{10} \\
&- \underset{1}{1}\cdot\underset{4}{4}\cdot\underset{3}{6}\cdot\underset{2}{4} - \underset{2}{1}\cdot\underset{1}{1}\cdot\underset{3}{6}\cdot\underset{4}{20} + \underset{2}{1}\cdot\underset{1}{1}\cdot\underset{4}{10}\cdot\underset{3}{10} + \underset{2}{1}\cdot\underset{3}{3}\cdot\underset{1}{1}\cdot\underset{4}{20} - \underset{2}{1}\cdot\underset{3}{3}\cdot\underset{4}{10}\cdot\underset{1}{1} \\
&- \underset{2}{1}\cdot\underset{4}{4}\cdot\underset{1}{1}\cdot\underset{3}{10} + \underset{2}{1}\cdot\underset{4}{4}\cdot\underset{3}{6}\cdot\underset{1}{1} + \underset{3}{1}\cdot\underset{1}{1}\cdot\underset{2}{3}\cdot\underset{4}{20} - \underset{3}{1}\cdot\underset{1}{1}\cdot\underset{4}{10}\cdot\underset{2}{4} - \underset{3}{1}\cdot\underset{2}{2}\cdot\underset{1}{1}\cdot\underset{4}{20} \\
&+ \underset{3}{1}\cdot\underset{2}{2}\cdot\underset{4}{10}\cdot\underset{1}{1} + \underset{3}{1}\cdot\underset{4}{4}\cdot\underset{1}{1}\cdot\underset{2}{4} - \underset{3}{1}\cdot\underset{4}{4}\cdot\underset{2}{3}\cdot\underset{1}{1} - \underset{4}{1}\cdot\underset{1}{1}\cdot\underset{2}{3}\cdot\underset{3}{10} + \underset{4}{1}\cdot\underset{1}{1}\cdot\underset{3}{6}\cdot\underset{2}{4} \\
&+ \underset{4}{1}\cdot\underset{2}{2}\cdot\underset{1}{1}\cdot\underset{3}{10} - \underset{4}{1}\cdot\underset{2}{2}\cdot\underset{3}{6}\cdot\underset{1}{1} - \underset{4}{1}\cdot\underset{3}{3}\cdot\underset{1}{1}\cdot\underset{2}{4} + \underset{4}{1}\cdot\underset{3}{3}\cdot\underset{2}{3}\cdot\underset{1}{1}.
\end{aligned}$$

It follows that the

* The symbol $n!$, called "n factorial" is an abbreviation for the continued product $1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)\, n$.

value of the given determinant is equal to $240 - 200 - 180 + 120 + 120 - 96 - 120 + 100 + 60 - 30 - 40 + 24 + 60 - 40 - 40 + 20 + 16 - 12 - 30 + 24 + 20 - 12 - 12 + 9 = 813 - 812 = +1$.

4. Exercises.

1. Determine the number of inversions in each of the following sequences of numbers:

$$\begin{array}{cccccccc} (a) & 5 & 2 & 4 & 7 & 3 & 1 & 6 \\ (b) & 3 & 6 & 1 & 5 & 4 & 7 & 2 \\ (c) & 7 & 6 & 4 & 5 & 3 & 2 & 1 \end{array}$$

2. How many terms are there in the expansion of a determinant of the 4th order? Of the 5th order? Of the 6th order?

3. Prove that the number of inversions in a row of numbers is not changed if all the numbers are increased or decreased by the same amount.

4. Show that if a row of integers is divided into two sections, such that all the numbers in the left section are less than any number in the right section, then the number of inversions in the original row is equal to the sum of the number of inversions in the left part and that in the right part.

5. Generalize the theorem of the preceding exercise so as to cover the case in which a row of integers is divided into more than two sections.

6. Determine the values of each of the following determinants by the method explained and illustrated in Section 3:

$$(a)\begin{vmatrix} 4 & -1 & 2 \\ 0 & 3 & -5 \\ -2 & 1 & 4 \end{vmatrix}; \quad (b)\begin{vmatrix} 5 & -3 & 2 \\ -4 & 1 & -2 \\ 3 & 5 & -2 \end{vmatrix}; \quad (c)\begin{vmatrix} 4 & -6 & 3 \\ -6 & 1 & 2 \\ 3 & 2 & 5 \end{vmatrix}.$$

7. Determine also the values of the following determinants:

$$(a)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 3 & -4 \\ 3 & 2 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{vmatrix}; \quad (b)\begin{vmatrix} 2 & -1 & 3 & -1 \\ 4 & -2 & -1 & 3 \\ 2 & -1 & -4 & 4 \\ 10 & -5 & -6 & 10 \end{vmatrix}; \quad (c)\begin{vmatrix} 2 & 2 & 2 & 10 \\ 0 & 0 & 1 & 2 \\ 3 & 4 & -3 & 2 \\ 1 & -2 & 4 & 5 \end{vmatrix};$$

$$(d)\begin{vmatrix} 2 & 3 & 1 & -1 \\ 2 & 0 & 0 & 3 \\ 4 & 1 & 0 & 1 \\ -1 & 2 & -2 & 1 \end{vmatrix}; \quad (e)\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix}.$$

5. Elementary Theorems. The determination of the value of a determinant by means of Definition V is quite laborious even for a determinant of order 4, as will have been discovered by the reader who has done all the parts of Exercise 7. For determinants whose order exceeds 4, this method becomes quite useless. Nevertheless it is important for the reader to do the exercises in the preceding set so that he may become thoroughly familiar with the content of Definition V. In the next few sections we shall derive

from this definition a chain of theorems which will supply us with the more useful methods for evaluating a determinant which will be employed in our further work.

THEOREM 1. **The interchange of two adjacent numbers in a row of integers which is an arrangement of the integers from 1 to n either increases or decreases the number of inversions of the row by one.**

Proof. Let p and q be adjacent. We have then to compare the number of inversions of the sets

$$(1) \ \ldots pq \ldots \qquad \text{and} \qquad (2) \ \ldots qp \ldots$$

Let us suppose $p < q$. The inversions which arise in (1) from any number preceding p or following q will also occur in (2), for all such numbers are followed by the same numbers in (2) as in (1). Furthermore p is followed in (2) by the same numbers which follow it in (1), except q; but since $q > p$ this change does not affect the number of inversions, so that the number of inversions due to p is the same in (2) as it was in (1). Finally q is followed in (2) by the same numbers as in (1) and moreover by p, which is less than q, so that the number of inversions due to q is one more in (2) than it was in (1). We conclude that there is a gain of one inversion in passing from (1) to (2), and consequently a loss of one inversion in passing from (2) to (1). Since in (2) the larger one of the two indices that are interchanged precedes the smaller one, whereas in (1) the condition is the opposite one, the proof of the theorem is complete.

THEOREM 2. **The interchange of any two numbers in a row of integers which is an arrangement of the integers from 1 to n changes the number of inversions in the row by an odd number.**

Proof. Let there be k numbers between p and q and let us compare the two arrangements

$$(3) \ \ldots p \overset{k}{\overline{\ldots}} q \ldots \qquad \text{and} \qquad (4) \ \ldots q \overset{k}{\overline{\ldots}} p \ldots .$$

By interchanging p successively with each of the k numbers which lie between p and q in (3), we obtain an arrangement which may be represented by

$$\ldots \ \overset{k}{\overline{\ldots}} \, p \, q \ldots ;$$

interchanging now q with p and with the k numbers preceding it, we obtain the arrangement (4). Thus we obtain (4) from (3) as the last of $2k + 1$ arrangements, each one of which is obtained from the preceding one by the interchange of two adjacent numbers. Hence the number of inversions in (4) is obtained, in virtue of Theorem 1, from that of (3) by $2k + 1$ changes of one, some of which are losses and the others gains. Since $2k + 1$ is odd, no matter what integer k is, the net result will be the loss or the gain of an odd number of inversions. For, if there are l losses and g gains, and $l > g$, the net result will be $l - g = l_1$ losses; but $l + g = 2k + 1$ and therefore $2l = l_1 + 2k + 1$, from which we conclude that l_1 is odd. And if the number of gains exceeds the number of losses it is shown in exactly similar fashion that the net result consists of an odd number of gains.

Remark. By allowing the change in the number of inversions to take negative as well as positive values we can say that every interchange of two numbers in a row of integers which is an arrangement of the integers from 1 to n changes the number of inversions by an odd number.

THEOREM 3. **If an arrangement of the integers $1 \dots n$ can be obtained from the natural order, or can be restored to the natural order, by an even (odd) number of interchanges of a pair of numbers, it will have an even (odd) number of inversions.**

Proof. The natural order presents no inversions and every interchange of a pair of numbers changes the number of inversions by an odd number. The truth of the theorem follows therefore from the fact that the sum of an even number of odd numbers is even, while the sum of an odd number of odd numbers is odd.

THEOREM 4. **The value of a determinant is not changed if the columns are made into rows and the rows into columns.**

Proof. Let the given determinant be $|a_{ij}|$, $i, j = 1, 2, \dots, n$; and let the determinant obtained by making the rows into columns and the columns into rows be designated by $|b_{ij}|$, $i, j = 1, 2, \dots, n$. Then $b_{ij} = a_{ji}$. An arbitrary term in the development of $|a_{ij}|$ has the form $a_{1c_1}a_{2c_2} \dots a_{nc_n}$ in which

$$(5) \quad c_1\, c_2 \dots c_n$$

is some arrangement of the integers $1, 2, \dots, n$; and the sign of this term depends on the number of inversions in the row of

numbers (5) (see Remark 2 on page 3). Moreover, this term is equal to

$$(6)\qquad b_{c_1 1}b_{c_2 2}\ldots b_{c_n n}$$

which is a term in the development of the determinant $|b_{ij}|$, except possibly for sign. In order to determine the sign of (6) we have to rearrange its factors so as to put the row indices in natural order; in doing this we shall put the column indices in irregular order and the sign of (6) will depend upon the number of inversions in that order. Now this order is obtained from the natural order by as many interchanges of pairs of numbers as it takes to restore the arrangement (5) to the natural order; consequently it will present an even or odd number of inversions according as the number of inversions in (5) is even or odd. Consequently the term (6) will appear in $|b_{ij}|$ with the same sign that the term $a_{1c_1}a_{2c_2}\ldots a_{nc_n}$ had in $|a_{ij}|$. But this last term was an arbitrary term in $|a_{ij}|$; therefore every term of the development of $|a_{ij}|$ occurs in the development of $|b_{ij}|$ and with the same sign. The same argument shows that the terms of $|b_{ij}|$ are all reproduced, in magnitude and in sign, in the development of $|a_{ij}|$. We have therefore proved that $|a_{ij}| = |b_{ij}|$.

COROLLARY. **If a theorem has been proved concerning the rows of a general determinant, we may conclude at once that a similar theorem holds for the columns; and vice versa.**

THEOREM 5. **The interchange of two columns (rows) of a determinant causes the value of the determinant to change sign.**

Proof. Let the given determinant be $|a_{ij}|$ and let the determinant obtained from it by interchanging the columns whose indices are c_1 and c_2 be designated by $|b_{ij}|$. Then from every term of the former determinant, we can obtain one of the latter by writing b's in place of a's and interchanging the column indices c_1 and c_2. It follows from Theorem 2 and Definition V that these two terms will be opposite in sign while equal in numerical value. Since moreover every term of $|b_{ij}|$ can be obtained in this manner, our theorem has been proved; the alternate form, indicated in the parentheses, follows by application of the Corollary of Theorem 4.

THEOREM 6. **If all the elements of a row (column) are multiplied by the same number, the value of the determinant is multiplied by that number.**

This theorem is an immediate consequence of Definition V.

THEOREM 7. **If two columns (rows) of a determinant are proportional, the value of the determinant is zero.**

Proof. Let us suppose first that the corresponding elements of the columns whose indices are c_1 and c_2, are equal. Let the value of the given determinant be A, and that of the determinant obtained from it by interchanging the columns of indices c_1 and c_2 be B. We conclude then from Theorem 5 that $B = -A$; and from the fact that the two columns which have been interchanged are identical, that $B = A$. Therefore $A = -A$ and hence $A = 0$. If now two columns of a determinant are proportional, its value is equal, on account of Theorem 6, to a factor of proportionality multiplied by the value of a determinant in which two columns are identical; its value is therefore also equal to zero.

THEOREM 8. **If the elements of a column (row) of a determinant are binomials, its value is equal to the sum of the values of the two determinants which agree with the given determinant in every element except that the particular column (row) concerned consists in one of them of the first terms of the binomials and in the other one of the second terms.**

This theorem is also an immediate consequence of Definition V; the proof is left to the reader (see Section 6). For a determinant of the third order the theorem asserts among other facts that

$$\begin{vmatrix} a_{11} & a_{12}+k_1 & a_{13} \\ a_{21} & a_{22}+k_2 & a_{23} \\ a_{31} & a_{32}+k_3 & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}.$$

THEOREM 9. **To the elements of any row (column) of a determinant may be added arbitrary multiples of the corresponding elements of any other row (column) without affecting the value of the determinant.**

Proof. It is a consequence of Theorem 8 that the value of the new determinant is equal to that of the given determinant plus the value of a determinant in which two rows (columns) are proportional. From this remark the present theorem follows by use of Theorem 7.

Remark 1. Theorems 7 and 9 are the first objectives of the chain of theorems we are developing. The latter enables us to derive from a given determinant another one which is equal to it in value but in which all the elements but one of some one row

or column are equal to zero; such a change materially reduces the labor involved in the evaluation of a determinant.

Remark 2. To abbreviate our terminology we shall speak of the "addition of one row or column of a determinant to another row or column" with the meaning "addition of the elements of one row (column) to the corresponding elements of another row (column)."

Examples.

1. To calculate the [illegible] the determinant $\begin{vmatrix} 4 & -7 & 3 \\ 1 & 5 & -2 \\ -3 & 6 & 2 \end{vmatrix}$ we use Theorem 9 as [illegible] o row 1 we add row 2 multiplied by -4; and to row 3 we add row 2 multiplied by 3. Thus we find that A is equal to the value of the determinant $\begin{vmatrix} 0 & -27 & 11 \\ 1 & 5 & -2 \\ 0 & 21 & -4 \end{vmatrix}$. This determinant can readily be evaluated by means of Definition V; in this way we find that $A = -(-27 \cdot 1 \cdot -4) + 11 \cdot 1 \cdot 21 = 123$.

2. To evaluate the determinant $\begin{vmatrix} 4 & 3 & 2 & 9 \\ 5 & 6 & -1 & 10 \\ -7 & 8 & 4 & 5 \\ 2 & -1 & 3 & 4 \end{vmatrix}$ we add column 3 to column 1; and we add column 2, multiplied by -1, to column 4; thus we obtain $\begin{vmatrix} 6 & 3 & 2 & 6 \\ 4 & 6 & -1 & 4 \\ -3 & 8 & 4 & -3 \\ 5 & -1 & 3 & 5 \end{vmatrix}$ in which there are two equal columns. By Theorem 7 the value of this determinant is zero; hence it follows from Theorem 9 that the value of the given determinant is also zero.

6. Exercises.

1. Show by an actual count of the inversions that the number of inversions is changed by an odd number when the numbers 2 and 8 are interchanged in the row 5 2 4 7 3 8 6 1.

2. Prove Theorem 1 for the case $p > q$ without assuming it for the case $p < q$.

3. Write out a detailed proof of the Corollary to Theorem 4.

4. Write out a detailed proof of Theorem 6.

5. Also for Theorem 8.

6. Illustrate Theorems 4, 5, 7, and 9 by means of determinants of the 3rd and 4th orders.

7. Evaluate each of the following determinants:

$$(a) \begin{vmatrix} -5 & 2 & 3 \\ 2 & 6 & 4 \\ -1 & 14 & 11 \end{vmatrix}, \quad (b) \begin{vmatrix} 1 & -1 & 2 & -2 \\ -3 & 2 & -2 & 3 \\ -1 & 3 & 1 & -3 \\ 2 & 1 & -1 & 2 \end{vmatrix}; \quad (c) \begin{vmatrix} 2 & -2 & 3 & 1 \\ -3 & 4 & 17 & -2 \\ 5 & -6 & -9 & 3 \\ 7 & 8 & 25 & -4 \end{vmatrix}.$$

8. Calculate the value of each of the following determinants:

$$(a)\ \begin{vmatrix} 4 & 3 & 2 & 3 \\ 5 & -2 & 1 & 0 \\ -1 & 4 & 3 & 1 \\ 2 & 1 & 4 & -3 \end{vmatrix}; \qquad (b)\ \begin{vmatrix} 1 & 2 & 3 & -1 & -2 \\ 2 & 1 & 3 & -2 & 1 \\ 0 & 1 & 2 & -2 & 1 \\ 0 & -1 & -1 & 2 & 1 \\ 0 & 2 & 3 & 1 & -4 \end{vmatrix}.$$

7. Minors and Cofactors. The following theorems will enable us to reduce still further the arithmetical work involved in the evaluation of a determinant. They will moreover furnish a basis for the application of determinants to the solution of systems of linear equations.

THEOREM 10. **The determinant obtained from a given determinant by shifting rows and columns in such a way as to bring a certain element in the upper left-hand corner, without changing the relative position of the rows and columns which do not contain this element, has a value equal to that of the given determinant or to its negative according as the sum of the row and column indices of this element is even or odd.**

Proof. Let us consider first the determinant $|a_{11}a_{22}a_{33}a_{44}|$ and let us call its value A. The upper left-hand corner of the determinant, now occupied by a_{11} will be called the "leading position." To form a new determinant in which a_{34} is in the leading position, while the relative order of the 1st, 2nd, and 4th rows, and also of the 1st, 2nd, and 3rd columns, remains unchanged, we interchange the 3rd row successively with the 2nd and 1st rows; in virtue of Theorem 5, this operation causes the value of the determinant to change its sign twice, so that we can write

$$\begin{vmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

Now we interchange the 4th column successively with the 3rd, 2nd, and 1st columns; this leads to the desired result at the cost of three changes of sign. Therefore, the new determinant

$$\begin{vmatrix} a_{34} & a_{31} & a_{32} & a_{33} \\ a_{14} & a_{11} & a_{12} & a_{13} \\ a_{24} & a_{21} & a_{22} & a_{23} \\ a_{44} & a_{41} & a_{42} & a_{43} \end{vmatrix},$$

in which a_{34} occupies the leading position while the rows and columns which do not contain a_{34} have the same relative order as in the original determinant, has a value equal to $-A$.

It should now be easy to understand the proof for the general case. To bring the element a_{ij} of the nth order determinant $|a_{ij}|$ into the leading position without affecting the relative order of the rows and columns which do not contain this element, we interchange the ith row successively with the $(i-1)$th, $(i-2)$th, . . . , 1st rows; then we interchange the jth column successively with the $(j-1)$th, $(j-2)$th, . . . , 1st columns. This is accomplished by means of $i - 1 + j - 1 = i + j - 2$ interchanges and therefore accompanied by $i + j - 2$ changes of sign in the value of the determinant. Consequently the determinant which we obtain finally will have a value equal to that of $|a_{ij}|$ or to its negative according as $i + j$ is even or odd; this proves the theorem.

Definition VI. **The *minor of an element* of a determinant of order n is the determinant of order $n - 1$ obtained by deleting the row and column in which this element stands.**

Definition VII. **The *cofactor* of an element of a determinant is equal to its minor or to the negative of its minor according as the sum of the row and column indices of the element is even or odd.**

Notation. The value of the cofactor of the element a_{ij} is designated by A_{ij}.

Theorem 11. **All the terms in the expansion of the determinant $|a_{ij}|$ which contain a particular element as a factor are obtained, in magnitude and in sign, by multiplying that element by its cofactor.**

Proof. For the element a_{11} this theorem is an immediate consequence of Definition V. Let us again denote the value of $|a_{ij}|$ and also the determinant itself by A. To determine the sum of all the terms in the development of A which contain the element a_{ij} as a factor, we consider the determinant A', obtained from A, as in Theorem 10, by putting a_{ij} in the leading position without affecting the relation of the rows and columns which do not contain this element. Then $A' = (-1)^{i+j}A$ and hence $A = (-1)^{i+j}A'$. Since a_{ij} occupies the leading position in A' the sum of the terms in A' which contain a_{ij} as a factor is obtained by multiplying a_{ij} by its cofactor in A'. But the cofactor of a_{ij} in A' is equal to its minor in A'; and the minor of a_{ij} in A' is the same as the minor of

this element in A', because the relative order of the rows and columns which do not contain the element a_{ij} has not been changed in the transition from A to A'. Therefore the sum of the terms in A' which contain the factor a_{ij} is obtained by multiplying this element by its minor in A. Moreover the terms in A which contain the factor a_{ij} are obtained by multiplying those in A' by $(-1)^{i+j}$. Therefore the sum of the terms in A which contain the factor a_{ij} is equal to $(-1)^{i+j} \times a_{ij} \times$ the minor of a_{ij} in $A = a_{ij} \times$ the cofactor of a_{ij} in $A = a_{ij}A_{ij}$.

THEOREM 12. **The value of a determinant is equal to the algebraic sum of the products obtained by multiplying the elements of any column (row) by their cofactors.**

Proof. In every term of the expansion of a determinant, there is one and only one factor from each column (row). If therefore we select a column (row) arbitrarily and take the sum of all the terms which contain any one of its elements as a factor, we shall obtain the value of the determinant. Hence the present theorem is an immediate consequence of Theorem 11.

THEOREM 13. **The algebraic sum of the products of the elements of any column (row) by the cofactors of the corresponding elements of *another* column (row) is equal to zero.**

Proof. We observe that the cofactors of the elements of any column (row) are not affected by changes made in that column (row). The cofactors of the elements $a_{1c_1}, a_{2c_1}, \ldots, a_{nc_1}$ in $|a_{ij}|$ are therefore the same as those of the elements $a_{1c_2}, a_{2c_2}, \ldots, a_{nc_2}$ in the first of the columns so designated in the determinant

$$\begin{vmatrix} a_{11} \ldots a_{1c_2} \ldots a_{1c_2} \ldots a_{1n} \\ a_{21} \ldots a_{2c_2} \ldots a_{2c_2} \ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1} \ldots a_{nc_2} \ldots a_{nc_2} \ldots a_{nn} \end{vmatrix}.$$

This determinant is obtained from $|a_{ij}|$ by replacing its column c_1 by its column c_2 (we are taking $c_1 < c_2$) and leaving everything else unchanged. On the one hand it follows from Theorem 7 that the value of this determinant is zero; on the other hand we conclude from Theorem 12 and the remark made at the opening of this proof, that its value is equal to

$$a_{1c_2}A_{1c_1} + a_{2c_2}A_{2c_1} + \ldots + a_{nc_2}A_{nc_1}.$$

We conclude therefore that

$$a_{1c_2}A_{1c_1} + a_{2c_2}A_{2c_1} + \ldots + a_{nc_2}A_{nc_1} = 0$$

whenever $c_1 \neq c_2$. This proves our theorem.

Theorems 12 and 13 are the final objectives of the chain of theorems which was started in Section 5. By means of Theorems 9 and 12, the value of any numerical determinant can be determined without an amount of arithmetical labor that is out of proportion to the order of the determinant; Theorem 7 can frequently be used to reduce this labor still further. There are usually several effective ways in which Theorem 9 can be used; by practice the reader will soon develop skill in applying it.

Examples.

1. To determine the value A of the determinant $\begin{vmatrix} 2 & -1 & 3 \\ 4 & -3 & 2 \\ -3 & 2 & 1 \end{vmatrix}$ we add row 1 multiplied by -3 to row 2; and row 1 multiplied by 2 to row 3. We find then, by use of Theorem 9 that

$$A = \begin{vmatrix} 2 & -1 & 3 \\ -2 & 0 & -7 \\ 1 & 0 & 7 \end{vmatrix}.$$

It follows from Theorem 12 that A is equal to the sum of the elements of the 2nd column, each multiplied by its cofactor; but this sum reduces to the product of the element -1 by its cofactor. The row index of this element is 1, its column index is 2; therefore its cofactor is equal to the negative of its minor. Therefore $A = (-1) \times - \begin{vmatrix} -2 & -7 \\ 1 & 7 \end{vmatrix} = -14 + 7 = -7$.

Remark. The calculation of the value of a determinant by means of Theorem 12, as illustrated in the above example, is frequently called "developing the determinant according to a column (row)." It reduces the evaluation of any determinant to that of one of the next lower order.

2. To evaluate the determinant $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$ (see Example 3, page 4), we add the first column multiplied by -1 in turn to the 2nd, 3rd, and 4th columns; thus we find that the value A of the given determinant is equal to that of the determinant $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 5 & 9 \\ 1 & 3 & 9 & 19 \end{vmatrix}$ and hence, by use of Theorem

12, to that of the third order determinant $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 9 \\ 3 & 9 & 19 \end{vmatrix}$. To evaluate this last determinant, we add to the 2nd and 3rd columns respectively the 1st column multiplied by -2 and by -3. It is then found that

$$A = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 3 & 3 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 10 \end{vmatrix} = 1\,.$$

8. Exercises.

1. Illustrate Theorem 10 by means of determinants of the 3rd order and of the 4th order.

2. Calculate the value of the determinant $\begin{vmatrix} 5 & -7 & 0 \\ 2 & 3 & -1 \\ 6 & 4 & 2 \end{vmatrix}$ by developing it, without previous reduction, according to the 1st row; also according to the 3rd column. Verify that the two results are equal.

3. Verify, in the determinant of Exercise 2, that the sum of the products of the elements of the 2nd column by the cofactors of the corresponding elements of the 1st column is equal to zero.

4. Evaluate each of the following determinants by the cofactor method:

$$(a)\ \begin{vmatrix} 5 & -2 & 3 \\ 4 & 1 & -2 \\ -1 & 3 & 2 \end{vmatrix};\quad (b)\ \begin{vmatrix} 17 & 8 & 3 \\ -5 & 7 & 2 \\ 6 & -11 & -4 \end{vmatrix};\quad (c)\ \begin{vmatrix} 2 & 3 & 4 & -1 \\ -1 & -2 & 5 & 2 \\ 6 & 1 & -2 & 3 \\ -4 & 5 & 1 & -3 \end{vmatrix}.$$

5. Calculate the values of the following determinants:

$$(a)\ \begin{vmatrix} 18 & 3 & 5 & -3 \\ 14 & 7 & -4 & 1 \\ 8 & -2 & 6 & 2 \\ -10 & 11 & 1 & 5 \end{vmatrix};\quad (b)\ \begin{vmatrix} -3 & 2 & \frac{1}{2} & 7 \\ -1 & \frac{5}{3} & -4 & 2 \\ -3 & 5 & 6 & 3 \\ 2 & 4 & -3 & -10 \end{vmatrix};\quad (c)\ \begin{vmatrix} 4 & -2 & 4 & 1 \\ -2 & 1 & -2 & -2 \\ 4 & -2 & 4 & \frac{3}{2} \\ 1 & -2 & \frac{3}{2} & 1 \end{vmatrix}.$$

6. Also of the following:

$$(a)\ \begin{vmatrix} 2 & 8 & 6 & 14 & 12 \\ 7 & 0 & 1 & 6 & -4 \\ 3 & -6 & 3 & -5 & 0 \\ 1 & 4 & 3 & 7 & 7 \\ -1 & 6 & 11 & 10 & 23 \end{vmatrix};\qquad (b)\ \begin{vmatrix} 2 & -3 & 4 & -2 & 5 \\ 3 & -4 & 3 & 2 & 13 \\ 3 & -4 & 4 & 1 & -2 \\ 4 & -5 & 3 & 6 & 4 \\ 5 & -6 & 3 & 9 & -2 \end{vmatrix}.$$

7. Show that the cofactor of the element a_{rs} in the determinant $|a_{ij}|$, $i, j = 1, 2, \ldots, n$, is equal to the determinant obtained from $|a_{ij}|$ by replacing the element a_{rs} by 1 and all the other elements in the rth row (or in the sth column, or in both) by zeros.

8. Prove that if the rows and columns of a determinant are shifted, as in Theorem 10, so as to bring the element a_{rs} into the leading position, then the cofactor of any element in the new determinant is equal to the product of the cofactor of this same element in the given determinant by $(-1)^{r+s}$. (*Hint:* Make use of the preceding exercise.)

9. Matrices. Rank of a Matrix. Before proceeding to the application of the theory of determinants to the solution of systems of linear equations, we shall introduce some further concepts, which, although perhaps not indispensable, will aid considerably not only in the solution of such systems but in all our further work.

DEFINITION VIII. **A *matrix* is a rectangular array of numbers. The numbers composing the array are called the *elements* of the matrix; the horizontal and vertical lines of the array are called respectively *rows* and *columns* of the matrix.**

Notation. In writing a matrix double vertical bars are placed on either side of the array. Large parentheses are sometimes used instead of the double vertical bars; we shall adhere to the former notation. For example

$$\begin{Vmatrix} 2 & 3 & -1 \\ -4 & -5 & 6 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 5 & 3 \\ 4 & -1 & 2 \\ -3 & 2 & -1 \\ 6 & -3 & 4 \end{Vmatrix}, \quad \text{and} \quad \begin{Vmatrix} 4 & 2 & -3 & 0 \\ \frac{5}{8} & \sqrt{7} & 6 & -2 \\ \sqrt{5} & -4 & \frac{2}{5} & -1 \\ 0 & -6 & 3 & 5 \end{Vmatrix}$$

are matrices.

Abbreviated notations, similar to those used for determinants (see Section 2) are also used for matrices; for example, $||a_{ij}||$, $i = 1, \ldots, 5$; $j = 1, \ldots, 4$ represents a matrix of 5 rows and 4 columns and $||a_{ij}||$, $i, j = 1, 2, \ldots, n$ represents a square matrix of n rows and n columns. We shall also designate a matrix by the single letters $\mathbf{a}$ or $\mathbf{b}$.

Remark. We emphasize the fact that a matrix is merely an array of numbers and that no numerical value is attached to it. In particular it is important to notice the difference between a square matrix and a determinant. Although both are square arrays of numbers, the latter has a number associated with it, namely, its "value," but the former has no number associated with it. A determinant whose elements are identical with the corresponding elements of a square matrix is called the "determinant of the matrix." We also speak in such a case of the "matrix of the determinant."

DEFINITION IX. **The *rank of a matrix* is a positive integer or zero, r, such that it is possible to form a determinant of order r whose value is different from zero and whose rows and columns are obtained from the rows and columns of the matrix, whereas it is not possible to form a determinant of order $r + 1$ which satisfies the same conditions.**

Remark. It is an immediate consequence of this definition that if the rank of a matrix is r, then the value of every determinant of order $r + 1$, $r + 2$, etc., whose rows and columns are formed from the rows and columns of the matrix, will be zero. And if the rank of a matrix is zero, all its elements are zero.

10. Complementary Minors. Elementary Transformation of Matrices.

DEFINITION X. **Any determinant whose rows and columns are formed from the rows and columns of a matrix (determinant) is called a minor of the matrix (determinant).**

The terms "two-rowed minor," "three-rowed minor," etc., which we shall have frequent occasion to use, should be clear without further explanation.

Remark. The definition of the rank of a matrix can now be put in the following form: **The rank of a matrix is an integer, r, positive or zero, such that the matrix has a non-vanishing r-rowed minor but no non-vanishing minor of order higher than r.**

DEFINITION XI. **A *principal minor* of a square matrix (determinant) is a minor formed by using rows and columns of equal indices only.**

DEFINITION XII. **If the rows and columns used in forming the minor M_2 of a square matrix (determinant) are those which were left unused in the formation of the minor M_1, then M_1 and M_2 are a pair of *complementary minors* of the matrix (determinant). Either is the complement of the other.**

DEFINITION XIII. **The *algebraic complement* of a minor of a square matrix (determinant) is equal to its complement multiplied by that power of -1 whose exponent is equal to the sum of the indices of the rows and columns used in the formation of the minor.**

Remark. The one-rowed principal minors of a square matrix (determinant) are the elements of its principal diagonal; the complement of a single element is its minor; the algebraic complement of a single element is its cofactor. (Compare Definitions VI and VII.)

Examples.

1. The determinants $\begin{vmatrix} a_{23} & a_{25} \\ a_{43} & a_{45} \end{vmatrix}$ and $\begin{vmatrix} a_{13} & a_{14} & a_{15} \\ a_{23} & a_{24} & a_{25} \\ a_{53} & a_{54} & a_{55} \end{vmatrix}$ are two-rowed and three-rowed minors respectively of the determinant $|a_{ij}|$, $i, j = 1, \ldots, 5$.

2. The determinants $\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{51} & a_{52} & a_{54} \end{vmatrix}$ and $\begin{vmatrix} a_{23} & a_{25} \\ a_{43} & a_{45} \end{vmatrix}$ are complementary

minors of the determinant $|a_{ij}|$, $i, j = 1, \ldots, 5$; and $\begin{vmatrix} a_{11} & a_{13} & a_{15} \\ a_{31} & a_{33} & a_{35} \\ a_{51} & a_{53} & a_{55} \end{vmatrix}$ is a principal minor of the same determinant.

3. The algebraic complement of the two-rowed minor $\begin{vmatrix} a_{23} & a_{25} \\ a_{43} & a_{45} \end{vmatrix}$ of the determinant $|a_{ij}|$, $i, j = 1, \ldots, 5$ is equal to $(-1)^{2+4+3+5} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{51} & a_{52} & a_{54} \end{vmatrix}$; it is therefore equal to the complement of this two-rowed minor.

4. The algebraic complement of the three-rowed minor $\begin{vmatrix} a_{12} & a_{14} & a_{15} \\ a_{32} & a_{34} & a_{35} \\ a_{62} & a_{64} & a_{65} \end{vmatrix}$ of the determinant $|a_{ij}|$, $i, j = 1, \ldots, 6$ is equal to $(-1)^{1+3+6+2+4+5} \begin{vmatrix} a_{21} & a_{23} & a_{26} \\ a_{41} & a_{43} & a_{46} \\ a_{51} & a_{53} & a_{56} \end{vmatrix}$ and therefore equal to the negative of its complement.

The operations upon the rows and columns of a determinant which were discussed in Section 5 may also be performed upon the rows and columns of a matrix. But since a matrix has no "value," the theorems on determinants which were there obtained will have no exact analogues; we shall however be interested in the effect of these operations upon the rank of the matrix. We introduce first the following Definition.

DEFINITION XIV. **An *elementary transformation* of a matrix is one which consists in performing upon it one of the following operations: the interchange of two rows (columns); the multiplication of all the elements of a row (column) by a non-vanishing multiplier; the addition to the elements of one row (column) of multiples of the corresponding elements of another row (column).**

Remark. It should be clear that if the matrix $\mathbf{b}$ is obtained from the matrix $\mathbf{a}$ by an elementary transformation, then the matrix $\mathbf{a}$ is obtainable from $\mathbf{b}$ by an elementary transformation.

THEOREM 14. **The rank of a matrix is not affected by an elementary transformation.**

Proof. Let us consider the matrix $\mathbf{a} = ||a_{ij}||$, $i, j = 1, \ldots, n$ and let us suppose that its rank is r; it will then contain at least one non-vanishing r-rowed minor, while every $(r + 1)$-rowed minor vanishes. The interchange of two rows or columns of $\mathbf{a}$ and the multiplication of the elements of a row or column of $\mathbf{a}$ by a

non-zero constant either have no effect whatever upon its minors, or else they will multiply a minor by a non-zero constant; in neither case will these operations kill off a non-vanishing minor of **a** nor bring a vanishing minor back to life. These operations will therefore leave the rank r of the matrix unchanged. If to the ith row of **a** we add k times the jth row, an $(r + 1)$-rowed minor of **a** will not be changed in value if it does not contain the ith row, nor if it contains both the ith and the jth rows. Let us suppose therefore that M is an $(r + 1)$-rowed minor of **a** which contains the ith row but not the jth row; and let us denote by M' the corresponding minor of the matrix **a**$'$ obtained from **a** by adding k times the jth row to the ith row. Then it follows from Theorem 8 that the value of M' is equal to the value of M plus k times the value of another $(r + 1)$-rowed minor of **a**; but since every $(r + 1)$-rowed minor of **a** vanishes, it follows from this that the value of M' is also zero. Consequently every $(r + 1)$-rowed minor of **a**$'$ vanishes, so that the rank of **a**$' \leqq r$; that is, the rank of a matrix is not increased by any elementary transformation. But then it follows from the remark preceding this theorem that the rank must remain unchanged. For if it were decreased then the elementary transformation which carries the new matrix back to the original would have to increase the rank; and we have just seen that this can not happen. The theorem has therefore been proved.

COROLLARY. **If a matrix a$'$ is derived from another matrix a by a succession of elementary transformations the ranks of the two matrices are equal.**

Remark. This theorem and its corollary can be used in the determination of the rank of a matrix in the same way as Theorems 5, 6, and 9 are used in the evaluation of a determinant.

11. Exercises.

1. Write out the minors of $\|a_{ij}\|$, $i, j = 1, \ldots, 6$ formed by using the following sets of rows and columns: $i = 1, 2, 5, j = 2, 3, 6$; $i = 2, 3, 4, 6$, $j = 2, 4, 5, 6$; $i = 3, 6, j = 3, 6$.

2. Determine the algebraic complements of each of the minors of Exercise 1.

3. Show that the rank of the matrix $\begin{Vmatrix} -1 & 2 & 3 & -5 \\ 3 & -4 & 5 & 2 \\ 5 & -6 & 13 & -1 \\ 0 & 2 & 14 & -13 \end{Vmatrix}$ is 2.

4. Determine the rank of each of the following matrices:

$$(a) \left\| \begin{matrix} -2 & 3 & 5 \\ 5 & -1 & -3 \\ 4 & 7 & 9 \end{matrix} \right\|; \qquad (b) \left\| \begin{matrix} 2 & 3 & -4 \\ 4 & 6 & -5 \\ 6 & 9 & -9 \end{matrix} \right\|;$$

$$(c) \left\| \begin{matrix} 2 & 3 & -4 \\ 4 & 6 & -8 \\ 6 & 9 & -12 \end{matrix} \right\|; \qquad (d) \left\| \begin{matrix} 1 & 0 & -3 & 5 \\ -2 & 3 & 2 & -4 \\ 7 & -6 & -13 & 25 \\ 1 & 0 & -3 & 3 \end{matrix} \right\|.$$

5. Prove that the algebraic complement of a principal minor of a square matrix is equal to its complement.

6. Prove that one of two complementary minors of a square matrix is a principal minor if and only if the other one is a principal minor.

7. Prove that if the minor M_1 of a square matrix is the algebraic complement of the minor M_2, then M_2 is also the algebraic complement of M_1.

12. The Laplace Development of a Determinant. In the remaining sections of this chapter we shall develop some further interesting and important properties of determinants. These properties will find application in the later chapters, but they are not needed for the solution of systems of linear equations. The reader can proceed therefore from this point immediately to Chapters II, III, IV, and V, returning to the remainder of Chapter I after he has completed these.

Our first objective is a generalization of the cofactor development of a determinant, discussed in Section 7.

LEMMA 1. **The determinant obtained from a given determinant by shifting the rows and columns in such a way as to bring a specified k-rowed minor in the upper left-hand corner without changing the relative position of the rows and columns not involved in this minor has a value equal to that of the given determinant or of its negative according as the sum of the indices of the rows and columns used in this minor is even or odd.**

Proof. Let the indices of the rows and columns used in the k-rowed minor under consideration be $r_1, r_2, \ldots, r_k$ and $c_1, c_2, \ldots, c_k$ respectively. To accomplish our purpose, we interchange the r_1th row successively with each of the $r_1 - 1$ rows which lie above it; next we interchange the r_2th row successively with each of the $r_2 - 2$ rows which lie above it but below the 1st row, the r_3th row with each of the $r_3 - 3$ rows which lie above it but below the 2nd row, etc., until we have interchanged the r_kth row with

each of the $r_k - k$ rows which lie above it but below the $(k-1)$th row. Thus the rows whose indices are $r_1, r_2, \ldots, r_k$ have been placed in the positions of the first k rows, while the relative position of the remaining rows has remained unchanged; and this has been done by means of $r_1 - 1 + r_2 - 2 + \ldots + r_k - k$ interchanges of rows, so that the determinant we have obtained has a value equal to that of the given determinant multiplied by $(-1)^{r_1+r_2+\cdots+r_k-k(k+1)/2}$. We proceed now to shift the columns whose indices are $c_1, c_2, \ldots, c_k$ in such a way as to bring them in the position of the first k columns without affecting the relative order of the remaining columns; it should be easy to see that this is accomplished by means of $c_1 - 1 + c_2 - 2 + \ldots + c_k - k$ interchanges of columns and therefore at the cost of $c_1 + c_2 + \ldots + c_k - k(k+1)/2$ changes of sign. The final result in which the specified k-rowed minor is in the upper left-hand corner and in which the rows and columns not occurring in this minor have the same relative order as in the given determinant has therefore a value equal to that of the given determinant multiplied by a power of -1 whose exponent is $r_1 + r_2 + \ldots + r_k + c_1 + c_2 + \ldots + c_k - k(k+1)$. But, no matter what integer k may be, $k(k+1)$ is always even. Consequently the value of the final determinant is equal to that of the given determinant if $r_1 + r_2 + \ldots + r_k + c_1 + c_2 + \ldots + c_k$ is even, and equal to its negative if this sum is odd.

Remark. For $k = 1$, this lemma and its proof reduce to Theorem 10 and its proof.

LEMMA 2. **All the terms in the development of a determinant which contain as a factor any term in the development of a specified k-rowed minor are obtained in the product of this minor by its algebraic complement; and this product contains nothing but such terms of the development of the determinant.**

Proof. We shall prove this proposition first for the principal k-rowed minor in the upper left-hand corner; and we shall denote this minor temporarily by A_k. The algebraic complement of this minor is equal to its complement (see Exercise 5, Section 11). An arbitrary term in the development of this minor is $(-1)^c a_{1c_1} a_{2c_2} \ldots a_{kc_k}$, where $c_1, c_2, \ldots, c_k$ is a permutation of the numbers $1, 2, \ldots, k$ and c is the number of inversions in this permutation; an arbitrary term in the development of its complement is

$(-1)^{\gamma} a_{k+1,\gamma_{k+1}} a_{k+2,\gamma_{k+2}} \ldots a_{n\gamma_n}$, where $\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_n$ represents an arbitrary permutation of the set of numbers $k+1, k+2, \ldots, n$ and γ is the number of inversions of this permutation.* The product of these two terms is $(-1)^{c+\gamma} a_{1c_1} a_{2c_2} \ldots a_{kc_k} a_{k+1,\gamma_{k+1}} a_{k+2,\gamma_{k+2}} \ldots a_{n\gamma_n}$. Since the numbers of the set $c_1, c_2, \ldots, c_k$ are all less than those of the set $\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_n$, it follows from Exercise 4, Section 4 that the number of inversions of the total set $c_1, c_2, \ldots, c_k, \gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_n$ is equal to $c + \gamma$; hence this product is a term in the development of the original determinant. If, on the other hand, $(-1)^d a_{1d_1} a_{2d_2} \ldots a_{nd_n}$ is a term in this development which contains as a factor a term of A_k, then its first k factors must be elements of A_k and therefore $d_1, d_2, \ldots, d_k$ and $d_{k+1}, d_{k+2}, \ldots, d_n$ must be permutations of the sets $1, 2, \ldots, k$ and $k+1, k+2, \ldots, n$ respectively. Hence $a_{1d_1} a_{2d_2} \ldots a_{kd_k}$ and $a_{k+1,d_{k+1}} a_{k+2,d_{k+2}} \ldots a_{nd_n}$ will be terms in the developments of A_k and of its complement respectively, and the numerical factors, $+1$ or -1, will be such that their product is equal to $(-1)^d$. Our lemma has been proved therefore for the principal minor A_k.

To prove it for an arbitrary k-rowed minor B_k formed from the rows and columns whose indices are $r_1, r_2, \ldots, r_k$ and $c_1, c_2, \ldots, c_k$ respectively, we form first, as in Lemma 1, the determinant in which B_k occupies the upper left-hand corner. Let us call this new determinant, and also its value, A'; then $A' = (-1)^{c_1+c_2+\cdots+c_k+r_1+r_2+\cdots r_k} A$ and the minor B_k of A goes over into the minor A_k' of A'. Moreover the complement of B_k in A is the same as the complement of A_k' in A'. In virtue of these facts and of the first part of this proof, we conclude that the sum of the terms in A which contain a term of B_k as a factor is equal to $(-1)^{c_1+c_2+\cdots c_k+r_1+r_2+\cdots r_k} \times A_k' \times$ the complement of A_k' in A' $= (-1)^{c_1+c_2+\cdots+c_k+r_1+r_2+\cdots+r_k} \times B_k \times$ the complement of B_k in $A = B_k \times$ the algebraic complement of B_k. This completes the proof of the lemma.

* In the development of the algebraic complement the sign of this term is determined by the number of inversions of the set of integers obtained from $\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_n$ by diminishing each of them by k; but it follows from Exercise 3, Section 4 that this new set of integers has the same number of inversions as the set $\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_n$.

Remark. The special case of this lemma which arises when $k = 1$ is identical with Theorem 11.

THEROEM 15. **The value of a determinant is equal to the algebraic sum of the products obtained by multiplying each of the k-rowed minors that can be formed from any k rows (columns) of the determinant by their algebraic complements.**

Proof. Let us consider the rows whose indices are $r_1, r_2, \ldots, r_k$. Every term in the development of the determinant will contain as a factor a product of k elements selected from these k rows, one from each; and every such product will be a term in the development of some one k-rowed minor whose row indices are $r_1, r_2, \ldots, r_k$. Hence we shall obtain the value of the determinant if we take the sum of all the terms which contain as a factor any term in the development of any one of these k-rowed minors. But, since it was shown in Lemma 2 that for a given k-rowed minor all such terms are found, and without any additional terms, in the product of this minor by its algebraic complement, we can conclude that the sum of the products of all the k-rowed minors formed from the k rows, which were selected, by their algebraic complements is equal to the value of the determinant.

Remark. The evaluation of a determinant by the method explained above is called the **Laplace development** of the determinant. For the case $k = 1$, it reduces to the development according to a row or column discussed in Theorem 12.

Example.

The determinant $\begin{vmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_2 & b_3 \\ -a_2 & -b_2 & 0 & c_3 \\ -a_3 & -b_3 & -c_3 & 0 \end{vmatrix}$ can be evaluated in a very convenient way by means of the Laplace development. If we use the first two columns, we find that its value is

$$\begin{vmatrix} 0 & a_1 \\ -a_1 & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c_3 \\ -c_3 & 0 \end{vmatrix} - \begin{vmatrix} 0 & a_1 \\ -a_2 & -b_2 \end{vmatrix} \times \begin{vmatrix} b_2 & b_3 \\ -c_3 & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_1 \\ -a_3 & -b_3 \end{vmatrix} \times \begin{vmatrix} b_2 & b_3 \\ 0 & c_3 \end{vmatrix}$$

$$+ \begin{vmatrix} -a_1 & 0 \\ -a_2 & -b_2 \end{vmatrix} \times \begin{vmatrix} a_2 & a_3 \\ -c_3 & 0 \end{vmatrix} - \begin{vmatrix} -a_1 & 0 \\ -a_3 & -b_3 \end{vmatrix} \times \begin{vmatrix} a_2 & a_3 \\ 0 & c_3 \end{vmatrix} + \begin{vmatrix} -a_2 & -b_2 \\ -a_3 & -b_3 \end{vmatrix} \times \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

$$= a_1^2c_3^2 - a_1a_2b_3c_3 + a_1a_3b_2c_3 + a_1b_2a_3c_3 - a_1b_3a_2c_3 + (a_2b_3 - a_3b_2)^2 = a_1^2c_3^2 + (a_2b_3 - a_3b_2)^2 - 2\,a_1c_3(a_2b_3 - a_3b_2) = (a_1c_3 - a_2b_3 + a_3b_2)^2.$$

13. Exercises.

1. Evaluate each of the following determinants by the Laplace development, using the first two rows:

$$(a)\begin{vmatrix} -2 & 5 & 3 & 0 \\ 3 & 4 & 1 & 0 \\ -5 & 2 & -4 & 3 \\ 1 & -3 & 0 & 2 \end{vmatrix};\quad (b)\begin{vmatrix} 0 & -1 & 7 & -4 \\ 3 & 5 & -2 & 1 \\ -6 & 3 & 0 & -5 \\ 4 & 2 & 0 & 3 \end{vmatrix};\quad (c)\begin{vmatrix} a_1 & b_1 & c_1 & 0 \\ d_1 & e_1 & f_1 & 0 \\ 0 & a_1 & b_1 & c_1 \\ 0 & d_1 & e_1 & f_1 \end{vmatrix}.$$

2. Prove that the sum of the products of the k-rowed minors formed from k columns of a determinant by the algebraic complements of the corresponding k-rowed minors of another set of k columns (that is, a set of k columns in which there is at least one column that was not among the columns of the first set) is equal to zero.

3. Evaluate each of the following determinants by Laplace's development:

$$(a)\begin{vmatrix} 3 & 0 & -5 & 7 \\ -6 & 0 & 4 & -2 \\ -3 & 1 & 6 & 8 \\ 4 & -2 & 5 & -3 \end{vmatrix};\qquad (b)\begin{vmatrix} -2 & 5 & 4 & 6 & 0 \\ 3 & 1 & -2 & 8 & 5 \\ 5 & -7 & 10 & 3 & 0 \\ -8 & 4 & -1 & 12 & -9 \\ -4 & 6 & 3 & -2 & 0 \end{vmatrix};$$

$$(c)\begin{vmatrix} 1 & 3 & -8 & 4 & 0 \\ 4 & -5 & 2 & 7 & -3 \\ -5 & 0 & 1 & 2 & 5 \\ 6 & 0 & 4 & -5 & 1 \\ -2 & 0 & -3 & 3 & -4 \end{vmatrix}.$$

4. Prove that the value of a determinant of order n, which contains for any integral value of k, a matrix of k rows (columns) and $n - k + 1$ columns (rows) of which every element is zero, is itself zero.

5. If a determinant of order n contains a matrix of k rows (columns) and $n - k$ columns (rows) of which every element is zero, the value of the determinant is equal to the product of the values of a single k-rowed minor and its algebraic complement.

6. Prove that the algebraic complement of a specified k-rowed minor of a determinant is equal to the determinant obtained from the given one by replacing by 1 the elements in the principal diagonal of this k-rowed minor and by 0 all the other elements of the rows (columns) from which the minor is formed.

7. Prove that if s is the sum of the row and column indices of a certain k-rowed minor of a determinant and if a new determinant is formed from the given one in which this k-rowed minor has been shifted to the upper left-hand corner by the method of Lemma 1, Section 12, then the cofactors of any element a_{ij} in the two determinants differ only by the factor $(-1)^s$. (*Hint:* Compare Section 8, Exercise 8.)

8. Prove that under the conditions of the preceding problem the algebraic complements of two corresponding k-rowed minors of the two determinants differ only by the factor $(-1)^s$.

14. The Product of Two Determinants. The Laplace development enables us to express the product of two determinants of order n as a determinant of order $2\,n$. We denote by

$$P = \begin{vmatrix} |a_{ij}| & O \\ I' & |b_{ij}| \end{vmatrix} \quad i, j = 1, 2, \ldots, n$$

the determinant of order $2\,n$ whose first n rows consist of the rows of $|a_{ij}|$, each extended by n zeros, while the last n rows consist of the rows of $|b_{ij}|$ preceded by an arbitrary square matrix of order n; it should be easy to see, particularly in view of Exercise 5, Section 13, that the value of P is equal to the product of the values of $|a_{ij}|$ and $|b_{ij}|$, that is, $P = A \cdot B$, where A and B designate the values of the determinants $|a_{ij}|$ and $|b_{ij}|$ respectively.*

We choose now for the matrix I' the n-rowed square matrix in which the elements in the principal diagonal are all equal to -1 and all the other elements are zero. We shall designate by $C_1, C_2, \ldots, C_n, C_{n+1}, \ldots, C_{2n}$ the matrices of one column and $2n$ rows each formed by the successive columns of P. Moreover we shall introduce an abbreviation, current in all mathematical writing and probably familiar to many readers, for a sum of terms which differ in subscript only; namely, we shall write, in general, $\sum_{k=1}^{n} u_k$ for the sum $u_1 + u_2 + \ldots u_n$.

We proceed now to apply Theorem 9 to change the form of the determinant P as follows: we replace C_{n+1} by $C_{n+1} + b_{11}C_1 + b_{21}C_2 + \ldots + b_{n1}C_n$, that is, we add to the $(n + 1)$th column of P each of the first n columns after having multiplied them by $b_{11}, b_{21}, \ldots, b_{n1}$ respectively. The $(n + 1)$th column will then consist of the following elements: $a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{n1}$, $a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2n}b_{n1}, \ldots, a_{n1}b_{11} + a_{n2}b_{21} + \ldots + a_{nn}b_{n1}, 0, 0, \ldots, 0$; or, using the abbreviation which has just been explained, of $\sum_{k=1}^{n} a_{1k}b_{k1}$, $\sum_{k=1}^{n} a_{2k}b_{k1}, \ldots, \sum_{k=1}^{n} a_{nk}b_{k1}, 0, 0, \ldots, 0$. Next we replace the column C_{n+2} by $C_{n+2} + b_{12}C_1 + b_{22}C_2 + \ldots + b_{n2}C_n$, the column C_{n+3} by $C_{n+3} + b_{13}C_1 + b_{23}C_2 + \ldots + b_{n3}C_n$, and so forth, until

* The reader is advised to write out in full the determinant P and to carry out the operations described concisely and with abbreviated notation in the following paragraphs.

we have replaced the last column of P, namely, C_{2n}, by $C_{2n} + b_{1n}C_1 + b_{2n}C_2 + \dots + b_{nn}C_n$. Thus we obtain the following result:

$$P = AB = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} & \sum a_{1k}b_{k1} & \sum a_{1k}b_{k2} & \dots & \sum a_{1k}b_{kn} \\ a_{21} & a_{22} & \dots & a_{2n} & \sum a_{2k}b_{k1} & \sum a_{2k}b_{k2} & \dots & \sum a_{2k}b_{kn} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & \sum a_{nk}b_{k1} & \sum a_{nk}b_{k2} & \dots & \sum a_{nk}b_{kn} \\ -1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 & 0 & 0 & \dots & 0 \end{vmatrix},$$

in which all the sums designated by the symbol $\sum$ are to be extended over the range $k = 1, \dots, n$. This determinant is developed by Laplace's development using the last n columns; in accordance with Exercise 5, Section 13, the result will be equal to the product of the n-rowed minor in the upper right-hand corner by its algebraic complement. Thus we find that

$$\begin{aligned} P = AB &= (-1)^{1+2+\dots+n+(n+1)+(n+2)+\dots+2n} \left|\sum a_{ik}b_{kj}\right| \cdot (-1)^n \\ &= (-1)^{2n(2n+1)/2+n} \left|\sum a_{ik}b_{kj}\right| = (-1)^{2n^2+2n} \left|\sum a_{ik}b_{kj}\right| \\ &= \left|\sum a_{ik}b_{kj}\right|, \; i, j = 1, 2, \dots, n. \end{aligned}$$

Thus we have shown that the product of the two nth order determinants $|a_{ij}|$ and $|b_{ij}|$ is equal to an nth order determinant in which the element in the ith row and jth column is equal to the sum $\sum_{k=1}^{n} a_{ik}b_{kj}$. This result is stated in the following theorem.

THEOREM 16. **The product of two determinants of order n is equal to a determinant of order n in which the element of the ith row and jth column is equal to the sum of the products of the elements in the ith row of the first factor by the corresponding elements of the jth column of the second factor.**

Remark. We know from Theorem 4 that the value of a determinant is not changed if the columns are taken as rows and the rows as columns. Hence if we rewrite A (or B, or both) in such a way as to interchange rows and columns and then determine their

product in accordance with Theorem 16, we obtain the following extension of this theorem:

The product of two nth order determinants is an nth order determinant whose element in the ith row and jth column is

(1) equal to the sum of the products of the elements in the ith row of the first by the corresponding elements of the jth column of the second; or

(2) equal to the sum of the products of the elements of the ith column of the first by the corresponding elements of the jth column of the second; or

(3) equal to the sum of the products of the elements in the ith row of the first by the corresponding elements in the jth row of the second; or

(4) equal to the sum of the products of the elements in the ith column of the first by the corresponding elements in the jth row of the second.

This statement is expressed symbolically by the following formula:

$$AB = \left|\sum_{k=1}^{n} a_{ik}b_{kj}\right| = \left|\sum_{k=1}^{n} a_{ki}b_{kj}\right| = \left|\sum_{k=1}^{n} a_{ik}b_{jk}\right| = \left|\sum_{k=1}^{n} a_{ki}b_{jk}\right|, i, j = 1, \ldots, n.$$

A condensed form of the statement is that determinants may be multiplied rows by columns, or columns by columns, or rows by rows, or columns by rows. In most cases we shall use the first of these methods for multiplying two determinants and, unless the contrary is explicitly stated, it is to be understood that this is the case.

Examples.

1. If $|a_{ij}|$, $i, j = 1, 2, 3$ is used to denote the product of the two determinants $\begin{vmatrix} 3 & -5 & 6 \\ 2 & 0 & -1 \\ -4 & 8 & 9 \end{vmatrix}$ and $\begin{vmatrix} -1 & 7 & -3 \\ 0 & 6 & 2 \\ 5 & -4 & 1 \end{vmatrix}$ then a_{23} is equal to the sum of the products of the elements in the second row of the first of these by the corresponding elements of the third column of the second; therefore $a_{23} = 2 \cdot (-3) + 0 \cdot 2 + (-1) \cdot 1 = -7$. Similarly we find $a_{32} = (-4) \cdot 7 + 8 \cdot 6 + 9 \cdot (-4) = -16$.

2. The product, rows by columns, of the two preceding determinants is

$$\begin{vmatrix} 27 & -33 & -13 \\ -7 & 18 & -7 \\ 49 & -16 & 37 \end{vmatrix} = \begin{vmatrix} 27 & -33 & -13 \\ -7 & 18 & -7 \\ 0 & 110 & -12 \end{vmatrix} = \begin{vmatrix} -1 & 39 & -41 \\ -7 & 18 & -7 \\ 0 & 110 & -12 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 39 & -41 \\ 0 & -255 & 280 \\ 0 & 110 & -12 \end{vmatrix} = 27{,}740.$$

Their product, columns by columns, is

$$\begin{vmatrix} -23 & 49 & -9 \\ 45 & -67 & 23 \\ 39 & 0 & -11 \end{vmatrix} = 4 \times \begin{vmatrix} 1 & 4 & -9 \\ -6 & 48 & 23 \\ 18 & -55 & -11 \end{vmatrix} = 4 \times \begin{vmatrix} 1 & 0 & 0 \\ -6 & 72 & -31 \\ 18 & -127 & 151 \end{vmatrix} = 27{,}740.$$

Their product, rows by rows, is

$$\begin{vmatrix} -56 & -18 & 41 \\ 1 & -2 & 9 \\ 33 & 66 & -43 \end{vmatrix} = \begin{vmatrix} -56 & -130 & 545 \\ 1 & 0 & 0 \\ 33 & 132 & -340 \end{vmatrix} = \begin{vmatrix} -56 & -130 & 545 \\ 1 & 0 & 0 \\ -23 & 2 & 205 \end{vmatrix} = 27{,}740.$$

And their product, columns by rows, is equal to

$$\begin{vmatrix} 23 & 4 & 3 \\ -19 & 16 & -17 \\ -40 & 12 & 43 \end{vmatrix} = 4 \times \begin{vmatrix} 0 & 1 & 0 \\ -111 & 4 & -29 \\ -109 & 3 & 34 \end{vmatrix} = 4 \times \begin{vmatrix} 111 & 29 \\ 109 & -34 \end{vmatrix}$$

$$= 4 \times \begin{vmatrix} 2 & 63 \\ 109 & -34 \end{vmatrix} = 27{,}740.$$

15. Exercises.

1. Multiply the determinants $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{vmatrix}$ and $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 4 & 9 \end{vmatrix}$

(1) rows by columns;
(2) columns by rows.

2. Multiply the determinants $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$ and $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 \\ 1 & 2 & 3 & 0 \end{vmatrix}$

(1) rows by rows;
(2) columns by columns.

3. Determine the square and the cube of the determinant $\begin{vmatrix} 2 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & -2 & 1 \end{vmatrix}$.

4. Form the products in each of the four possible ways of two general second-order determinants and show that the results are equal in value.

5. Prove that a two-rowed minor of the product of two determinants is equal to a sum of products of two-rowed minors of the two determinants.

6. Prove a theorem for three-rowed minors similar to the theorem of Exercise 5.

7. Prove the general theorem of which the theorems of Exercises 5 and 6 are special cases.

8. Prove that the rank of the matrix of a product of two determinants can not exceed the rank of the matrix of either factor.

16. The Adjoint of a Determinant.

Definition XV. **The determinant of which the element in the ith row and jth column ($i, j = 1, \ldots, n$) is equal to the value of the co-**

factor A_{ij} of the element a_{ij} of the determinant $|a_{ij}|$, $i, j = 1, \ldots, n$ is called the *adjoint* of that determinant.

We shall denote by A and by A' the values of the determinant $|a_{ij}|$ and its adjoint respectively. By means of Theorems 16, 12, and 13 we should be able to see readily that the product rows by rows of $|a_{ij}|$ and its adjoint forms a determinant in which the elements of the principal diagonal are all equal to A, and the remaining elements are all zero; hence that $A \cdot A' = A^n$ and therefore that $A' = A^{n-1}$, if $A \neq 0$. This result gives us the following theorem.

THEOREM 17. **If the value of the determinant $|a_{ij}|$, $i, j = 1, \ldots, n$ is different from zero, then the value of the adjoint of this determinant is equal to the $(n - 1)$th power of the value of $|a_{ij}|$.**

The theorem which we have just proved is a special case of the following more general theorem.

THEOREM 18. **The value of a k-rowed minor of the adjoint of a determinant is equal to the product of the value of the algebraic complement of the corresponding k-rowed minor of the given determinant by the $(k - 1)$th power of the value of the given determinant, provided this latter value is different from zero.**

Proof. We will prove this theorem first for the k-rowed principal minor in the upper left-hand corner of the adjoint. We begin by forming the product of $|a_{ij}|$ by the determinant

$$\begin{vmatrix} A_{11} & A_{21} & \cdots & A_{k1} & 0 & 0 & \cdots & 0 \\ A_{12} & A_{22} & \cdots & A_{k2} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{1k} & A_{2k} & \cdots & A_{kk} & 0 & 0 & \cdots & 0 \\ A_{1,\,k+1} & A_{2,\,k+1} & \cdots & A_{k,\,k+1} & 1 & 0 & \cdots & 0 \\ A_{1,\,k+2} & A_{2,\,k+2} & \cdots & A_{k,\,k+2} & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{kn} & 0 & 0 & \cdots & 1 \end{vmatrix}.$$

The Laplace development of this determinant which uses the first k rows, together with Theorem 4, shows that its value is equal to that of the k-rowed principal minor which we wish to determine (compare Exercise 5, Section 13); let us denote this value by V_k. The product of $|a_{ij}|$ by this determinant is formed in accordance

with Theorem 16; if we make use again of Theorems 12 and 13, we obtain the following result:

$$A \cdot V_k = \begin{vmatrix} A & 0 & \dots & 0 & a_{1,\,k+1} & a_{1,\,k+2} & \dots & a_{1n} \\ 0 & A & \dots & 0 & a_{2,\,k+1} & a_{2,\,k+2} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A & a_{k,\,k+1} & a_{k,\,k+2} & \dots & a_{kn} \\ 0 & 0 & \dots & 0 & a_{k+1,\,k+1} & a_{k+1,\,k+2} & \dots & a_{k+1,\,n} \\ 0 & 0 & \dots & 0 & a_{k+2,\,k+1} & a_{k+2,\,k+2} & \dots & a_{k+2,\,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & a_{n,\,k+1} & a_{n,\,k+2} & \dots & a_{nn} \end{vmatrix}$$

If the determinant on the right-hand side of this equation is developed by Laplace's development, using the first k columns, we find, by thinking once more of Exercise 5, Section 13, that it reduces to $A^k \begin{vmatrix} a_{k+1,\,k+1} & \dots & a_{k+1,\,n} \\ \dots & \dots & \dots \\ a_{n,\,k+1} & \dots & a_{nn} \end{vmatrix}$; hence, since $A \neq 0$ by hypothesis, we conclude that

$$V_k = \begin{vmatrix} a_{k+1,\,k+1} & \dots & a_{k+1,\,n} \\ \dots & \dots & \dots \\ a_{n,\,k+1} & \dots & a_{nn} \end{vmatrix} \times A^{k-1}.$$

Now the determinant on the right of this equation is the algebraic complement of the minor of $|a_{ij}|$ which corresponds to the minor of its adjoint which we are having under consideration; the theorem has therefore been proved for this special case.

Let us now consider an arbitrary k-rowed minor M_k' of the adjoint; let M_k be the corresponding minor of $|a_{ij}|$ and let m_k be its algebraic complement. Let us furthermore denote by s_k the sum of the row and column indices used in M_k (and therefore in M_k'). We know then from Lemma 1, Section 12 that the value of the determinant obtained from $|a_{ij}|$ by a shifting of rows and columns which brings the minor M_k into the upper left-hand corner without altering the relative order of the rows and columns not involved in this minor is equal to $(-1)^{s_k}A$; and from Exercise 7, Section 13 we infer that the cofactors of the elements of this rearranged determinant differ from the cofactors of the same elements in $|a_{ij}|$ by the factor $(-1)^{s_k}$. Therefore, if $\overline{M}_k'$ denotes the k-rowed principal minor in the upper left-hand corner of the adjoint of the

rearranged determinant, every element of $\overline{M}_k'$ is equal to $(-1)^{s_k}$ times the corresponding element of M_k' and hence $\overline{M}_k' = (-1)^{ks_k}M_k'$. To the minor $\overline{M}_k'$ we can apply the conclusion reached in the first part of the present proof; the algebraic complement of the corresponding k-rowed minor in the rearranged determinant $|a_{ij}|$ is identical with the complement of M_k and hence equal to $(-1)^{s_k}m_k$. We have therefore the following result: $(-1)^{ks_k}M_k' = (-1)^{s_k}m_k \times [(-1)^{s_k}A]^{k-1}$, from which we conclude that $M_k' = m_kA^{k-1}$; this is the relation asserted in our theorem.

Remark. For $k = n$, Theorem 18 reduces to Theorem 17; for $k = 1$, it merely asserts that every element of the adjoint is equal to the cofactor of the corresponding element of the given determinant $|a_{ij}|$. If we denote by α_{ij} the cofactor of the element A_{ij} of the adjoint, we obtain by putting $k = n - 1$, the following corollary.

COROLLARY. **The cofactor α_{ij} of the element A_{ij} in the adjoint of the determinant $|a_{ij}|$ is equal to $a_{ij}A^{n-2}$.**

17. The Derivative of a Determinant. The elements of the determinants whose properties we have been discussing have been constants. If these elements are functions of a single variable t, let us call them $u_{ij}(t)$, then the value of the determinant is also a function of this variable. Denoting this function by the symbol $U(t)$, we have

$$U(t) = |u_{ij}(t)|, \; i, j = 1, \ldots, n.$$

We inquire now for a convenient form in which to write the derivative of $U(t)$ with respect to t. To obtain such a form, we recall two facts:

(1) That $U(t)$ is the sum of terms $\pm u_{1c_1}(t)u_{2c_2}(t) \ldots u_{nc_n}(t)$, in which $c_1, c_2, \ldots, c_n$ are successively the different permutations of the set of integers, $1, 2, \ldots, n$ and the sign of the term depends on the character of the permutation;

(2) That the derivative of a product of two or more functions is equal to the sum of all the products obtainable from the given product by replacing one factor at the time by its derivative; for example, if $'$ denotes differentiation with respect to t, $(u_1u_2u_3)' = u_1'u_2u_3 + u_1u_2'u_3 + u_1u_2u_3'$.

From these facts we conclude that $U'(t)$ is equal to the sum of the sums $\sum \pm u_1{}'_{c_1}u_{2c_2} \ldots u_{nc_n}$, $\sum \pm u_{1c_1}u_2{}'_{c_2} \ldots u_{nc_n}$, $\ldots$,

$\sum \pm u_{1c_1} u_{2c_2} \ldots u'_{nc_n}$, in each of which $c_1, c_2, \ldots, c_n$ are successively the different permutations of the set of integers $1, 2, \ldots, n$ and the sign of the term is plus or minus according as the number of inversions in the permutation is even or odd. But then it follows from (1) that the first of these sums is the expansion of a determinant obtainable from $|u_{ij}(t)|$ by replacing the elements in its first row by their derivatives; also, that the second sum is the expansion of the determinant obtainable from $|u_{ij}(t)|$ by replacing the elements in the second row by their derivatives; and so forth. We can therefore state the following answer to our inquiry.

THEOREM 19. **The derivative with respect to t of the determinant $U(t) = |u_{ij}(t)|$, $i, j = 1, \ldots, n$ is equal to the sum of the n determinants obtained from $U(t)$ by replacing the elements of one row (column) at the time by their derivatives.**

Examples.

1. The adjoint of the determinant $\begin{vmatrix} 2 & -1 & 3 \\ 0 & 1 & -2 \\ 1 & -3 & 2 \end{vmatrix}$ is found to be $\begin{vmatrix} -4 & -2 & -1 \\ -7 & 1 & 5 \\ -1 & 4 & 2 \end{vmatrix}$.

The value, A, of the first determinant is -9; and the value, A', of the adjoint is $81 = (-9)^2$. The cofactor of the element in the 3rd row and 2nd column of the adjoint is $-\begin{vmatrix} -4 & -1 \\ -7 & 5 \end{vmatrix}$; its value is 27, which is also the product of the corresponding element in the original determinant, -3, by the 1st power of the value of the determinant, -9.

2. The derivative of the value of the determinant

$$\begin{vmatrix} t^2 - 3t & 2t & 2t^2 - 7t + 4 \\ 3t^2 + 1 & 4t^2 & 6t - 3 \\ 2t + 1 & 2t^2 + 6t - 3 & t^2 + 8t - 7 \end{vmatrix}$$

is equal to the value of the sum

$$\begin{vmatrix} 2t - 3 & 2t & 2t^2 - 7t + 4 \\ 6t & 4t^2 & 6t - 3 \\ 2 & 2t^2 + 6t - 3 & t^2 + 8t - 7 \end{vmatrix} + \begin{vmatrix} t^2 - 3t & 2 & 2t^2 - 7t + 4 \\ 3t^2 + 1 & 8t & 6t - 3 \\ 2t + 1 & 4t + 6 & t^2 + 8t - 7 \end{vmatrix}$$
$$+ \begin{vmatrix} t^2 - 3t & 2t & 4t - 7 \\ 3t^2 + 1 & 4t^2 & 6 \\ 2t + 1 & 2t^2 + 6t - 3 & 2t + 8 \end{vmatrix}.$$

18. Exercises.

1. Determine the adjoints of each of the following determinants:

(a) $\begin{vmatrix} 1 & -4 & 5 \\ -4 & 5 & 1 \\ 5 & 1 & -4 \end{vmatrix}$; (b) $\begin{vmatrix} 1 & 0 & -1 & 3 \\ 0 & 4 & 2 & -1 \\ -1 & 2 & -2 & 4 \\ 3 & -1 & 4 & 1 \end{vmatrix}$; (c) $\begin{vmatrix} 1 & 2 & -2 & 1 \\ 2 & 1 & 3 & -4 \\ 4 & 5 & -1 & -2 \\ 3 & 0 & 8 & -9 \end{vmatrix}$.

2. Differentiate each of the following determinants:

$$(a)\ \begin{vmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{vmatrix};\quad (b)\ \begin{vmatrix} t^2 - 1 & -t^2 + 4 \\ 1/(t-2) & 1/(t+1) \end{vmatrix};\quad (c)\ \begin{vmatrix} a_1 \cos t & b_1 \sin t & c_1 \\ a_2 \sin t & b_2 \cos t & c_2 \\ c_1 & c_2 & 0 \end{vmatrix}.$$

3. Determine the 1st, 2nd, and 3rd derivatives of the determinant

$$\begin{vmatrix} a_{11} - t & a_{12} & a_{13} \\ a_{21} & a_{22} - t & a_{23} \\ a_{31} & a_{32} & a_{33} - t \end{vmatrix}.$$

4. A symmetric determinant being defined as one in which, for every pair of indices i and j, $a_{ij} = a_{ji}$, show that the adjoint of a symmetric determinant is itself symmetric.

5. Show that the value of the adjoint of the determinant

$$\begin{vmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{vmatrix}$$

is zero and the rank of its matrix is 1.

6. Prove that the value of the determinant

$$\begin{vmatrix} a_1t + p_1 & a_1t + p_2 & a_1t + p_3 \\ a_2t + q_1 & a_2t + q_2 & a_2t + q_3 \\ a_3t + r_1 & a_3t + r_2 & a_3t + r_3 \end{vmatrix}$$

is a function of the first degree in t.

7. Verify, by direct computation, the Corollary to Theorem 18 for the case of 3rd order determinants.

8. Work out a formula for the 2nd derivative of the determinant $|u_{ij}(t)|$, $i, j = 1, \ldots, n$.

19. Miscellaneous Exercises.

1. Determine the value of each of the following determinants:

$$(a)\ \begin{vmatrix} -4 & 2 & 3 \\ 5 & -3 & 2 \\ 7 & 1 & -6 \end{vmatrix};\quad (b)\ \begin{vmatrix} 4 & -1 & 3 & 2 \\ 12 & 5 & -7 & 4 \\ -3 & 6 & 5 & -9 \\ 14 & 15 & 9 & -10 \end{vmatrix};\quad (c)\ \begin{vmatrix} 3 & 2 & -1 & -2 \\ 2 & 0 & 1 & 0 \\ -1 & 1 & 3 & 2 \\ -2 & 0 & 2 & 0 \end{vmatrix}.$$

2. Determine the rank of each of the following matrices:

$$(a)\ \begin{Vmatrix} 2 & -3 & 5 & 6 \\ 1 & 2 & 4 & -5 \\ 4 & -13 & 7 & 28 \end{Vmatrix};\quad (b)\ \begin{Vmatrix} -3 & 2 & 1 & -2 \\ 2 & -1 & 3 & 4 \\ 0 & 1 & 11 & 8 \\ 1 & 7 & 0 & -5 \end{Vmatrix};\quad (c)\ \begin{Vmatrix} 4 & -2 & 4 & 1 \\ -2 & 1 & -2 & -2 \\ 4 & -2 & 4 & \frac{3}{2} \\ 1 & -2 & \frac{3}{2} & 1 \end{Vmatrix}.$$

3. Compute the adjoints of the determinants (a) and (c) in Exercise 1.

4. Determine, by inspection, the sum of the values of the determinants

$$\begin{vmatrix} 4 & 2 & 3 \\ -1 & -3 & 2 \\ 6 & 1 & -6 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} -3 & 2 & 3 \\ 1 & -3 & 2 \\ -6 & 1 & -6 \end{vmatrix}.$$

5. Show that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (a-b)(b-c)(c-d)(a-c)(b-d)(a-d).$

6. Determine the value of the determinant $\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix}$.*

7. Show that the square of the value of the determinant in Exercise 5 is equal to the value of the determinant $\begin{vmatrix} 4 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{vmatrix}$, where $s_k = a^k + b^k + c^k + d^k$, for $k = 1, \dots, 6$.

8. Using the notation s_k, introduced in the preceding exercise, set up an nth order determinant which is equal in value to the square of the value of the determinant in Exercise 6.

9. Show that if $\omega^3 = 1$, $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{vmatrix} = 0$. What is the rank of the matrix of this determinant?

10. Prove that if the elements a_{ij} of a determinant $|a_{ij}|$ are independent variables then the partial derivative of its value with respect to a particular element is equal to the cofactor of this element.

11. Prove that the second partial derivative of the value of a determinant $|a_{ij}|$ with respect to the variables a_{ij} and a_{rs} is equal to the algebraic complement of the two-rowed minor $\begin{vmatrix} a_{ij} & a_{is} \\ a_{rj} & a_{rs} \end{vmatrix}$.

12. Prove that the kth partial derivative of the value of the determinant $|a_{ij}|$ with respect to the elements $a_{r_1c_1}, a_{r_2c_2}, \dots, a_{r_kc_k}$ of which no two have the same row-index nor the same column-index, is equal to the algebraic complement of the k-rowed minor whose rows and columns have the indices $r_1, r_2, \dots, r_k$ and $c_1, c_2, \dots, c_k$ respectively.

* A determinant of this form is frequently referred to as a Vandermonde determinant.

CHAPTER II

LINEAR EQUATIONS

20. Definition and Notation.

DEFINITION I. **An *equation* in one or more variables is called *homogeneous* if, after the right-hand side has been reduced to zero, the terms on the left-hand side are all of the same degree in all the variables jointly.**

Remark. It follows from this definition that a linear equation (that is, an equation of the first degree in all the variables jointly) is homogeneous if and only if it contains no term independent of the variables.

Notation. We shall be dealing with systems of equations in n variables; the variables will be designated by $x_1, x_2, \ldots, x_n$. A linear equation will therefore have the form:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = k.$$

It will be homogeneous if and only if $k = 0$.

A system of linear equations will be written in the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= k_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= k_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{p1}x_1 + a_{p2}x_2 + \ldots + a_{pn}x_n &= k_p. \end{aligned}$$

The first subscript in each coefficient designates the equation in which it occurs; the second subscript indicates the variable which the coefficient multiplies. It will be a system of homogeneous equations if and only if $k_1 = k_2 = \ldots = k_p = 0$.

It is convenient to designate the entire system of equations briefly by writing

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = k_i, \; i = 1, 2, \ldots, p.$$

The coefficients $a_{i_1}, a_{i_2}, \ldots, a_{in}, i = 1, 2, \ldots, p$ form a matrix of p rows and n columns; this matrix will be called the **coefficient matrix** (abbreviated c.m.) of the system of equations. If $p = n$, this matrix will be a square matrix; the corresponding

determinant is $|a_{ij}|$, $i, j = 1, \ldots, n$, which will be called the **coefficient determinant** of the system, and its value will be indicated, as in Chapter I, by A. If we write not merely the coefficients of the variables, but also the known terms which appear on the right-hand sides of the equations, we obtain a matrix of p rows and $n + 1$ columns; this matrix will be called the **augmented matrix** (abbreviated a.m.) of the system of equations. It will be a square matrix if and only if $p = n + 1$, that is, if the number of equations in the system is one greater than the number of variables.

21. The System of n Linear Non-homogeneous Equations in n variables.

Theorem 1. **If the coefficient determinant of a system of n linear equations in n variables has a value A, different from zero, then the system has a single solution, consisting of one value for each of the variables; these are equal to fractions whose denominators are all equal to A, and whose numerators are the values of the determinants obtained from the coefficient determinant by replacing the coefficient of each variable in turn by the known terms as they appear on the right-hand side of the equations.**

Proof. Let the equations be written in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= k_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= k_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= k_n. \end{aligned} \tag{1}$$

If they are multiplied by the cofactors $A_{11}, A_{21}, \ldots, A_{n1}$ of the elements $a_{11}, a_{21}, \ldots, a_{n1}$ in the first column of the coefficient determinant and then added, it follows from Theorems 12 and 13 of Chapter I that the coefficient of x_1 in the sum is equal to A, while the coefficients of $x_2, x_3, \ldots, x_n$ are all zero. Therefore we find that

$$A\,x_1 = k_1A_{11} + k_2A_{21} + \cdots + k_nA_{n1}.$$

If we use as multipliers the cofactors $A_{12}, A_{22}, \ldots, A_{n2}$ of the elements in the second column of the coefficient determinant and add, we find that

$$A\,x_2 = k_1A_{12} + k_2A_{22} + \cdots + k_nA_{n2};$$

and by the use of the cofactors of the elements in the third, fourth, $\ldots$, nth columns as multipliers, we obtain $n - 2$ further equa-

tions of the same general form. They may be written simultaneously in the form:

$$(2)\ A\,x_i = k_1A_{1i} + k_2A_{2i} + \ldots + k_nA_{ni},\ i = 1, 2, \ldots, n.$$

Any set of values of the variables $x_1, x_2, \ldots, x_n$ which satisfy equations (1) must satisfy these conditions. Since $A \neq 0$, there is one and only one value for each x_i; and if we recall the observation made at the opening of the proof of Theorem 13 of Chapter I (see page 13), we will recognize that the right-hand side of (2) is the expansion according to its ith column of the determinant obtained from the coefficient determinant by replacing its ith column by the constants $k_1, k_2, \ldots, k_n$ on the right-hand sides of the given equations. Consequently, if $A \neq 0$, the system of equations (1) can not have more than one solution, namely, the one given by the values

$$(3)\quad x_i = \frac{k_1A_{1i} + k_2A_{2i} + \ldots + k_nA_{ni}}{A} = \frac{\sum_{j=1}^{n} k_jA_{ji}}{A},\ i = 1, 2, \ldots, n.$$

It remains to show that the values given by (3) actually do satisfy the equations (1). Substitution of these values in the left-hand side of the rth equation of this system gives, by repeated use of the abbreviated notation for sums introduced in Section 14 (see page 25)

$$\frac{\sum_{i=1}^{n} a_{ri} \sum_{j=1}^{n} k_jA_{ji}}{A} = \frac{\sum_{j=1}^{n} k_j \sum_{i=1}^{n} a_{ri}A_{ji}}{A}.$$

But it follows from Theorems 12 and 13 that $\sum_{i=1}^{n} a_{ri}A_{ji}$ is equal to zero if $j \neq r$ and equal to A if $j = r$. Hence the only term in the sum which is different from zero is the one obtained for $j = r$, so that it reduces to k_rA and the values given by (3) do therefore actually satisfy the given equations. This completes the proof of the theorem.

Remark. The rule given by this theorem for writing down at once the solution of a system of n linear equations in n variables whose coefficient determinant has a value different from zero, is known as **Cramer's rule.**

22. The System of n Linear Homogeneous Equations in n Variables. From the result obtained in the preceding section, we derive some important consequences.

THEOREM 2. **A system of n linear homogeneous equations in n variables whose coefficient determinant has a value different from zero, possesses the solution which consists of zero for each of the variables,** ***and no others.***

Proof. It should be clear by inspection that, if $k_1 = k_2 = \cdots = k_n = 0$, the equations of the system (1) are satisfied by the values $x_1 = x_2 = \cdots = x_n = 0$. That the system has no other solution if $A \neq 0$ follows from the proof of Theorem 1.

Remark. It should be evident that every system of p linear homogeneous equations in n variables $x_1, x_2, \ldots, x_n$ possesses the solution $x_1 = x_2 = \cdots = x_n = 0$; this solution of such a system is called the **trivial solution.**

On account of the frequent use to be made of it in the later parts of this book (see Sections 41 and 82) we state the following corollary which is an immediate consequence of the preceding theorem.

COROLLARY. **In order that a system of n linear homogeneous equations in n variables shall have a non-trivial solution, it is necessary that the value of its coefficient determinant shall be zero.**

23. The System of $n + 1$ Linear Non-homogeneous Equations in n Variables.

THEOREM 3. **A system of $n+1$ linear equations in n variables, whose coefficient matrix has rank n, possesses a solution if and only if the determinant of its augmented matrix has the value zero; and in this case the solution is unique.**

Proof. Let the equations be written in the form:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = k_i,\ i = 1, 2, \ldots, n + 1.$$

Since the c.m. is of rank n, we can find in it at least one determinant of order n, which has a non-zero value. And, because the order in which the equations that constitute the system are written is clearly a matter of indifference, we can suppose without loss of generality that this determinant is the coefficient determinant of the first n equations of the system. These equations possess therefore a single solution, which can be written down

according to Cramer's rule; consequently the entire system of $n + 1$ equations can not have more than one solution. It will possess one solution if the values of the variables determined by the first n equations also satisfy the $(n + 1)$th equation. Now it is possible to express these values in terms of the cofactors of the elements in the last row of the a.m. If we use the familiar capital letter notation to designate the cofactors of the elements of the a.m., we find from the first n equations, since $K_{n+1} \neq 0$:

$$K_{n+1}x_i = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,\,i-1} & k_1 & a_{1,\,i+1} & \cdots & a_{1,\,n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,\,i-1} & k_2 & a_{2,\,i+1} & \cdots & a_{2,\,n-1} & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{n,\,i-1} & k_n & a_{n,\,i+1} & \cdots & a_{n,\,n-1} & a_{nn} \end{vmatrix}$$

$$= (-1)^{n-i} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,\,i-1} & a_{1,\,i+1} & \cdots & a_{1,\,n-1} & a_{1n} & k_1 \\ a_{21} & a_{22} & \cdots & a_{2,\,i-1} & a_{2,\,i+1} & \cdots & a_{2,\,n-1} & a_{2n} & k_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{n,\,i-1} & a_{n,\,i+1} & \cdots & a_{n,\,n-1} & a_{nn} & k_n \end{vmatrix}.$$

The latter of these determinants is obtained from the former by interchanging its ith column successively with each of the $n - i$ columns which follow it. But the last written determinant is clearly the minor of the element $a_{n+1,\,i}$ in the a.m. and therefore equal to $(-1)^{n+1+i}$ times the cofactor $A_{n+1,\,i}$ of this element. We conclude therefore that

$$K_{n+1}x_i = (-1)^{n-i+n+1+i} A_{n+1,\,i} = -A_{n+1,\,i},\ i = 1, 2, \ldots, n.$$

Now we substitute the values of x_i obtained in this way in the last equation of the system. We find then that the system possesses a solution if and only if $a_{n+1,\,1}A_{n+1,\,1} + a_{n+1,\,2}A_{n+1,\,2} + \cdots + a_{n+1,\,n}A_{n+1,\,n} + k_{n+1}K_{n+1} = 0$. But this is, in virtue of Theorem 12, Chapter I, the condition that the determinant of the augmented matrix be equal to zero. We have therefore proved the theorem.

COROLLARY. **A system of $n + 1$ linear homogeneous equations in n variables whose coefficient matrix has rank n possesses only the trivial solution.**

Examples.

1. The system of equations $3x - 2y = 4$, $2x + 3y = 5$, $x - y = 2$ has no solution. For, while the c.m. is clearly of rank 2 (the value of the determinant $\begin{vmatrix} 3 & -2 \\ 2 & 3 \end{vmatrix}$ is different from zero), the determinant of the augmented

matrix has a value different from zero; this determinant is $\begin{vmatrix} 3 & -2 & 4 \\ 2 & 3 & 5 \\ 1 & -1 & 2 \end{vmatrix}$ and its value is 11.

2. Let us consider the system of equations $3x - 2y + z = 7$, $2x + 3y - 4z = -9$, $x - y + z = 4$, $x + 2y + 3z = 5$. The rank of the c.m. is 3; and the determinant formed of the coefficients of the first three equations, namely, $\begin{vmatrix} 3 & -2 & 1 \\ 2 & 3 & -4 \\ 1 & -1 & 1 \end{vmatrix}$ has the value $+4$. The determinant of the a.m. is $\begin{vmatrix} 3 & -2 & 1 & 7 \\ 2 & 3 & -4 & -9 \\ 1 & -1 & 1 & 4 \\ 1 & 2 & 3 & 5 \end{vmatrix}$. It should not be difficult to show that its value is zero. This being done, it follows from Theorem 3 that the system has a single solution, which may be found by solving the first three equations of the system by Cramer's rule. Thus we find, by use of Theorem 1, that

$$x = \frac{\begin{vmatrix} 7 & -2 & 1 \\ -9 & 3 & -4 \\ 4 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 1 \\ 2 & 3 & -4 \\ 1 & -1 & 1 \end{vmatrix}} = \frac{4}{4} = 1, \quad y = \frac{\begin{vmatrix} 3 & 7 & 1 \\ 2 & -9 & -4 \\ 1 & 4 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 1 \\ 2 & 3 & -4 \\ 1 & -1 & 1 \end{vmatrix}} = \frac{-4}{4} = -1, \quad z = \frac{\begin{vmatrix} 3 & -2 & 7 \\ 2 & 3 & -9 \\ 1 & -1 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -2 & 1 \\ 2 & 3 & -4 \\ 1 & -1 & 1 \end{vmatrix}} = \frac{8}{4} = 2.$$

24. Exercises.

1. Give illustrations of systems of homogeneous and of non-homogeneous linear equations in 2, 3, and 4 variables.

2. Determine the rank of each of the following matrices:

$$(a) \begin{Vmatrix} 3 & -4 & 5 & 2 \\ -1 & 2 & 5 & 1 \\ 5 & -6 & 19 & 5 \end{Vmatrix}; \quad (b) \begin{Vmatrix} 5 & -2 & 3 \\ -4 & 10 & 6 \\ 15 & -6 & 9 \end{Vmatrix}; \quad (c) \begin{Vmatrix} 3 & -2 & 5 & 1 \\ -1 & 4 & 3 & 2 \\ 7 & -8 & 7 & 0 \\ 1 & 6 & 16 & 4 \end{Vmatrix}.$$

3. Solve each of the following systems of equations by Cramer's rule:

(*a*) $3x + 4y - 2z = 5$, $4x + 3y + 8z = -4$, $2x + 8y - 3z = 5$.

(*b*) $x + 2y + 3z = 4$, $3x - 2y - v = 6$, $2x - 3z - 3v = 0$, $y - 4v = 15$.

4. Proceed in the same way with the following systems of equations:

(*a*) $3x + y - 4u = 9$, $-5y + 3z + 2u = 18$, $x - 6y + 7z = 33$, $-2x - 8y + 5z + 2u = 18$.

(*b*) $3x + y + z = 20$, $x + 4y + 3v = 30$, $6x + z + 3v = 40$, $8y + 3z + 5v = 50$.

5. Determine, for each of the following systems of equations, whether they possess a solution; solve those systems for which a solution exists:

(*a*) $2x - y + 1 = 0$, $x + 2y + 2 = 0$, $15x + 20y + 24 = 0$.

(*b*) $x - y + 4 = 0$, $3x + 2y + 3 = 0$, $x + 4y - 5 = 0$.

(*c*) $5x - 3y - 7 = 0$, $x + 2y + 4 = 0$, $3x - 7y + 1 = 0$.

6. Also for each of the following systems:

(a) $6x + 5y - 12z = -11,\ 5x - 2y - 4z = 2,\ 3x - 3y - 6z = 0,$ $x + 18y - 8z = -24.$

(b) $2x + 3y + 3z = 1,\ 3x - y - 4z = 4,\ -2x + 3y + 7z = -3,$ $-4x + 11y + 21z = -9.$

(c) $3x - y + 2z = -3,\ -2x + 2y - 3z = 2,\ 5x + 3y - 2z = -4,$ $4x + 6y - 6z = 3.$

25. The System of $n - 1$ Linear Homogeneous Equations in n Variables.

THEOREM 4. **If the rank of the coefficient matrix of a system of $n - 1$ linear homogeneous equations in n variables is equal to $n - 1$, the system has a single infinitude of solutions; the ratios of the variables are equal to the ratios of the cofactors of the elements in the nth row of the determinant obtained by writing a row of arbitrary elements $p_1, p_2, \ldots, p_n$ under the coefficient matrix.**

Proof. We write the equations in the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0,\ i = 1, 2, \ldots, n - 1.$$

Since the c.m. is of rank $n - 1$, it contains a non-vanishing determinant of order $n - 1$, and there is no loss of generality if we suppose that this is the determinant formed by the coefficients of the variables $x_1, x_2, \ldots, x_{n-1}$. In virtue of Theorem 1, the equations possess therefore a single solution for $x_1, x_2, \ldots, x_{n-1}$ for every definite value assigned to x_n; hence there is a single infinitude of solutions.

We can apply Cramer's rule to solve the equations for $x_1, x_2, \ldots, x_{n-1}$ in terms of x_n; we find

$$P_n x_i = -x_n \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1,i-1} & a_{1n} & a_{1,i+1} & \ldots & a_{1,n-1} \\ a_{21} & a_{22} & \ldots & a_{2,i-1} & a_{2n} & a_{2,i+1} & \ldots & a_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,i-1} & a_{n-1,n} & a_{n-1,i+1} & \ldots & a_{n-1,n-1} \end{vmatrix}$$

$$= (-1)^{n-i}x_n \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1,i-1} & a_{1,i+1} & \ldots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2,i-1} & a_{2,i+1} & \ldots & a_{2,n-1} & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,i-1} & a_{n-1,i+1} & \ldots & a_{n-1,n-1} & a_{n-1,n} \end{vmatrix}$$

$= x_n P_i$, $i = 1, 2, \ldots, n$, where $P_1, P_2, \ldots, P_n$ denote the cofactors of the arbitrary elements $p_1, p_2, \ldots, p_n$ in the determinant obtained from the c.m. by writing a row of arbitrary elements

under it. If we write this result in the form of a continued proportion, we obtain the result stated in the theorem, namely:

$$x_1 : x_2 : \ldots : x_n = P_1 : P_2 : \ldots : P_n.$$

COROLLARY. **If the rank of the coefficient matrix of a system of p linear homogeneous equations in n variables is $n - 1$, the system has a single infinitude of solutions; the ratios of the variables are equal to the ratios of the cofactors of the elements in some row of an n-rowed minor of the coefficient matrix.**

Proof. If the hypothesis is satisfied, it is clear that we must have $p \geqq n - 1$ and that there must be at least one set of $n - 1$ among the p equations which satisfies the conditions of Theorem 4. The order in which the equations of the system are written is immaterial and we may therefore suppose that the first $n - 1$ equations of the system furnish one such set. We will show now that the solutions of this set also satisfy the remaining equations of the original system of p equations. For, if we substitute kP_i, $i = 1, 2, \ldots, n$ for x_i in the rth equation ($r = n, n + 1, \ldots, p$), the left-hand side becomes $k(a_{r1}P_1 + a_{r2}P_2 + \cdots + a_{rn}P_n)$; but the expression in parentheses is the expansion of the nth order determinant obtained by writing the coefficients of the rth equation under the c.m. of the first $n - 1$ equations, so that its value is equal to zero. Moreover the set of the first $n - 1$ equations has, in virtue of Theorem 4, no other solutions besides those of the single infinitude indicated above; consequently solutions which are obtained by using another set of $n - 1$ equations selected from the given system must be contained among this single infinitude. Hence our corollary is proved.

26. The Adjoint of a Vanishing Determinant. Symmetric Determinants. By means of Theorem 4 we are able to obtain a valuable extension of Theorems 17 and 18 of Chapter I. For in these theorems we had to make the hypothesis that the value of the determinant under consideration was different from zero; it is this restriction which we are now able to remove. We begin by proving the following theorem.

THEOREM 5. **If the value of a determinant is zero, the rank of the matrix of the adjoint of the determinant is equal to 0 or 1.**

Proof. If the cofactor of every element of the determinant vanishes, the rank of the matrix of the adjoint is clearly equal to

zero; if this is not the case, let us suppose that $A_{kr} \neq 0$. Then for every i the equations

$$\begin{array}{l} a_{11}A_{i1} + a_{12}A_{i2} + \cdots + a_{1n}A_{in} = 0 \\ a_{21}A_{i1} + a_{22}A_{i2} + \cdots + a_{2n}A_{in} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{k-1,1}A_{i1} + a_{k-1,2}A_{i2} + \cdots + a_{k-1,n}A_{in} = 0 \\ a_{k+1,1}A_{i1} + a_{k+1,2}A_{i2} + \cdots + a_{k+1,n}A_{in} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}A_{i1} + a_{n2}A_{i2} + \cdots + a_{nn}A_{in} = 0 \end{array}$$

hold, in virtue of Theorems 12 and 13 of Chapter I. But these equations may be looked upon as a system of $n - 1$ linear homogeneous equations in the n variables $A_{i1}, A_{i2}, \ldots, A_{in}$; and since the coefficient matrix surely contains the determinant A_{kr}, its rank is $n - 1$. We can therefore apply Theorem 4 and we find that

$$A_{i1} : A_{i2} : \ldots : A_{in} = A_{k1} : A_{k2} : \ldots : A_{kn}, \text{ for } i = 1, 2, \ldots, n.$$

This means that the different rows of the adjoint are proportional, so that every two-rowed minor vanishes; the rank of the adjoint is therefore less than 2.

Corollary. **The conclusions of Theorems 17 and 18 of Chapter I still hold when the value of the determinant $|a_{ij}|$ is zero.**

We shall now prove a few important consequences of this theorem which refer to symmetric square matrices (usually called symmetric determinants) and which are needed in our later work (see Chapter VIII).

Definition II. **A *symmetric square matrix* is a matrix $\|a_{ij}\|$, $i, j = 1, 2, \ldots, n$, in which, for every i and every j, $a_{ij} = a_{ji}$.***

Definition III. **A *singular square matrix* is one whose determinant vanishes.**

Theorem 6. **If all the $(n - 1)$-rowed principal minors of a singular symmetric square matrix vanish, its rank is less than $n - 1$.**

Proof. We have to show that under the hypothesis every element of the adjoint vanishes. Now it should be easy to see that the adjoint of the given symmetric square matrix is itself symmetric, so that $A_{ij} = A_{ji}$ for every i and j.* Moreover it follows

* Compare Exercise 4, Section 18.

from Theorem 5 that every two-rowed minor of the adjoint has the value zero, that is, $\begin{vmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{vmatrix} = 0$ for every i and j. Hence, $A_{ii}A_{jj} = A_{ij}^2$. But $A_{ii} = A_{jj} = 0$, by hypothesis. Consequently $A_{ij} = 0$; and this is what our theorem asserts.

COROLLARY. **If the rank of a singular symmetric square matrix of order n is $n - 1$, then it contains** ***at least one non-vanishing $(n-1)$-rowed principal minor.***

THEOREM 7. **All the $(n - 1)$-rowed principal minors of a singular symmetric square matrix which do not vanish have the same sign.**

Proof. If A_{ii} and A_{jj} are two non-vanishing principal minors, then it follows from the hypothesis, as in the proof of Theorem 6, that $A_{ii}A_{jj} = A_{ij}^2 > 0$; hence A_{ii} and A_{jj} have the same sign.

COROLLARY. **If the sum of the $(n - 1)$-rowed principal minors of a singular symmetric square matrix is equal to zero, the rank of the matrix is less than $n - 1$.**

This corollary follows immediately from Theorems 7 and 6.

27. The System of n Linear Non-homogeneous Equations in n Variables, *continued.* In Section 21 we have seen that a system of n linear non-homogeneous equations in n variables, whose c.m. has rank n possesses a single solution; this solution may be determined by means of Cramer's rule. We return now to such a system of equations but under the hypothesis that the rank of the c.m. is $n - 1$, and we shall prove the following theorem.

THEOREM 8. **If the coefficient matrix of a system of n linear non-homogeneous equations in n variables is of rank $n - 1$, the system will have a single infinitude of solutions or no solution, according as the rank of the augmented matrix is equal to or greater than $n - 1$.**

Proof. We write the equations in the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = k_i, \quad i = 1, 2, \ldots, n.$$

Since the rank of the c.m. is $n - 1$, there is at least one set of $n - 1$ of the equations and at least one set of $n - 1$ of the variables, such that the c.m. of these variables in this particular set of equations is of rank $n - 1$; these equations can therefore be solved by Cramer's rule for $n - 1$ of the variables in terms of the nth variable, as soon as a value has been assigned to this variable. Con-

sequently this set of $n-1$ equations possesses a single infinitude of solutions. It remains to determine whether these solutions will also satisfy the single remaining equation. This question can be answered by means of Theorem 3.

If the special set of $n-1$ variables consists of $x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots x_n$, the determinant of the augmented matrix of the related system of n equations in $n-1$ variables is

$$\begin{vmatrix} a_{11} & a_{12} \ldots a_{1,j-1} & a_{1,j+1} \ldots a_{1n} & k_1 - a_{1j}x_j \\ a_{21} & a_{22} \ldots a_{2,j-1} & a_{2,j+1} \ldots a_{2n} & k_2 - a_{2j}x_j \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} \ldots a_{n,j-1} & a_{n,j+1} \ldots a_{nn} & k_n - a_{nj}x_j \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} \ldots a_{1,j-1} & a_{1,j+1} \ldots a_{1n} & k_1 \\ a_{21} \ldots a_{2,j-1} & a_{2,j+1} \ldots a_{2n} & k_2 \\ \cdots & \cdots & \cdots \\ a_{n1} \ldots a_{n,j-1} & a_{n,j+1} \ldots a_{nn} & k_n \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} \ldots a_{1n} \\ \cdots\cdots \\ a_{n1} \ldots a_{nn} \end{vmatrix} (-1)^{n-j+1} x_j.$$

But since the rank of the c.m. of the given system is $n-1$, the last term vanishes; therefore the determinant of the augmented matrix of the related system reduces to the first term, which is an n-rowed minor of the augmented matrix of the given system. If the rank of this matrix is $n-1$, then every one of its n-rowed minors vanishes; hence the augmented matrix of the related system has rank $n-1$ and we conclude, by use of Theorem 3, that all the solutions of the special set of $n-1$ equations also satisfy the nth equation. But if the rank of this matrix is n, then, for some one of the variables x_j, the determinant of the augmented matrix of the related system will be different from zero, no matter what value is assigned to x_j; and in this case we conclude, again by use of Theorem 3, that the given system of equations does not possess a solution. This completes the proof of the theorem.

Remark. The theorems proved in Sections 21, 22, 23, 25, and 27 are special cases of a more general theorem which asserts that a system of p linear equations in n variables possesses one or more solutions if and only if the ranks of the c.m. and the a.m. are equal. A proof of this theorem may be found in the books referred to in

Section 1. The special cases dealt with here suffice for the applications to be made in the later chapters of this book; they furnish moreover a suitable introduction to the study of the more general cases. For this reason and also in order to avoid too elaborate algebraic discussions, we have restricted ourselves to these special cases. The reader is urged to make himself thoroughly familiar with the content and the proofs of these theorems; they will repeatedly be referred to in our further work.

Examples.

1. The system of equations

$$2x - 3y + 5z = 0, \quad 3x + 2y - z = 0, \quad 4x + 7y - 7z = 0$$

determines the ratios of the variables x, y, and z. For the value of the coefficient determinant $\begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -1 \\ 4 & 7 & -7 \end{vmatrix}$ is readily found to be zero. Since the two-rowed minor in the upper right-hand corner is different from zero, the rank of the c.m. is 2. Consequently, we conclude from Theorem 4 and its corollary that the ratios of the variables are equal to the ratios of the cofactors of the elements in the last row. Thus we find that

$$x : y : z = \begin{vmatrix} -3 & 5 \\ 2 & -1 \end{vmatrix} : -\begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} : \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = -7 : 17 : 13.$$

It is easily verified that any three numbers which have these ratios satisfy the given equations.

2. To determine the ratios of x, y, and z from the system of equations

$$x - 2y + 3z = 0, \quad -3x + 6y + z = 0$$

we observe first of all that the rank of the c.m. is 2. In view of Theorem 4 we can conclude that the ratios of the variables are equal to the ratios of the cofactors of the elements in the third row of the determinant $\begin{vmatrix} 1 & -2 & 3 \\ -3 & 6 & 1 \\ p_1 & p_2 & p_3 \end{vmatrix}$; hence we find that

$$x \cdot y \,.\, z = \begin{vmatrix} -2 & 3 \\ 6 & 1 \end{vmatrix} : -\begin{vmatrix} 1 & 3 \\ -3 & 1 \end{vmatrix} : \begin{vmatrix} 1 & -2 \\ -3 & 6 \end{vmatrix} = -20 : -10 : 0 = 2 : 1 : 0.$$

3. In virtue of Theorem 8, the system of equations

$$2x - 3y + 5z = 1, \quad 3x + 2y - z = 4, \quad 4x + 7y - 7z = 5$$

has no solution. For it should be easy to show that the rank of the c.m. is 2; and since the value of the determinant $\begin{vmatrix} 2 & -3 & 1 \\ 3 & 2 & 4 \\ 4 & 7 & 5 \end{vmatrix}$ formed from the a.m. is -26, the rank of the a.m. is 3.

4. In the system of equations

$$2x - 3y + 5z = 1, \quad 3x + 2y - z = 4, \quad 4x + 7y - 7z = 7$$

the ranks of the c.m. and the a.m. are both equal to 2. It follows therefore from Theorem 8 that the system possesses a single infinitude of solutions which may be found by solving two of the equations for two of the variables in terms of the third. If we solve the first two equations for x and y in terms of z, we find that

$$x = \frac{14 - 7z}{13} \quad \text{and} \quad y = \frac{5 + 17z}{13}.$$

It is readily verified that these expressions satisfy the three given equations for all values of z.

28. Exercises.

1. Determine the ratios of the variables from each of the following systems of homogeneous equations:

(*a*) $x + 3y - z = 0,\ -2y + z = 0,\ 5x + y + 2z = 0.$

(*b*) $4x + 6y + 3z - 84v = 0,\ 2x + y + 3z - 48v = 0,\ -2x + y + z - 12v = 0,\ 4x + 4y - z - 24v = 0.$

(*c*) $2x - y + 2z = 0,\ -x + 2y + 2z = 0,\ 2x + 2y - z = 0.$

2. Proceed similarly with each of the following systems:

(*a*) $2x - y + 2z = 0,\ x + 2y + 2z = 0.$

(*b*) $5x - y + 3z = 0,\ 10x - 2y + 4z = 0.$

(*c*) $2x - y + 2z + 5u = 0,\ 3x + y - z + 2u = 0,\ 4x - 2y + 3u = 0.$

3. Show that none of the following systems of equations possess a solution:

(*a*) $3x - 4y = 5,\ 2x + 5y = 3,\ 6x - 8y = 4.$

(*b*) $2x + 3y - 4z = 3,\ x - 2y + 3z = 1,\ 4x + y - 2z = -2,\ 2x - 4y + 5z = 3.$

(*c*) $5x - y + 2z = 12,\ 2x + 3y = 7,\ 3x - 2y - 4z = -2,\ 4x + y - 2z = 5.$

4. Determine which of the following systems possess a solution; solve those for which a solution exists:

(*a*) $x + 6y - 2z = 3,\ 4x + 2y + z = 1,\ 2x + 5z = -5.$

(*b*) $4x + 2y + z = 1,\ x + y - 2z = 3,\ 3x + y + 3z = -4.$

(*c*) $2x - y + 3z = 5,\ 4x - 3y + 2z = -4,\ y + 4z = 10.$

5. Determine the conditions under which the system of equations

$$a_1x + b_1y + c_1z + d_1 = 0, \quad a_2x + b_2y + c_2z + d_2 = 0,$$
$$a_3x + b_3y + c_3z + d_3 = 0, \quad a_4x + b_4y + c_4z + d_4 = 0,$$

possess one solution.

6. Determine the ranks of the adjoints of the following square matrices:

$$(a)\ \begin{Vmatrix} -1 & 2 & 1 & -2 \\ 2 & 3 & -2 & -3 \\ 1 & -2 & -1 & 2 \\ -2 & -3 & 2 & 4 \end{Vmatrix}; \qquad (b)\ \begin{Vmatrix} -1 & 2 & 1 & -2 \\ 2 & 3 & -2 & -3 \\ 1 & -2 & -1 & 2 \\ -2 & -3 & 2 & 3 \end{Vmatrix}.$$

7. Prove:

(*a*) If the rank of the matrix $\begin{Vmatrix} p & u & v \\ u & q & w \\ v & w & r \end{Vmatrix}$ is 2, then the values of no two of the expressions $pq - u^2$, $qr - w^2$ and $rp - v^2$ are opposite in sign.

(*b*) If $\begin{vmatrix} p & u & v \\ u & q & w \\ v & w & r \end{vmatrix} = 0$ and $pq + qr + rp - u^2 - v^2 - w^2 = 0$, the rank of the matrix in (*a*) is 1.

8. Prove that if the system of equations

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = k_i, \quad i = 1, 2, \ldots, n$$

in which the value of the coefficient determinant $|a_{ij}|$ is different from zero, is solved for $x_1, x_2, \ldots, x_n$ in terms of $k_1, k_2, \ldots, k_n$, then the value of the determinant formed by the coefficients of $k_1, k_2, \ldots, k_n$ is equal to the reciprocal of the value of the determinant $|a_{ij}|$.

CHAPTER III

POINTS AND LINES

The primary object of Solid Analytical Geometry is the study of the geometry of three-dimensional space by algebraic methods. This end is accomplished by means of coördinate systems or frames of reference. Such systems enable us, as in Plane Analytical Geometry, to determine algebraic entities corresponding to various geometric elements. We start this study with the simplest geometric element, the point in three-space.

29. The Cartesian Coördinates of a Point in Three-space. The simplest frame of reference is furnished by three mutually perpendicular planes, called the **coördinate planes.** It is customary to take one of these planes horizontal, the other two vertical.

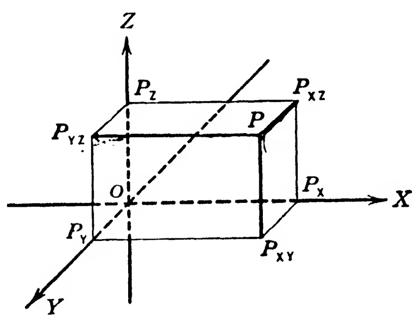

FIG. 1

The point common to the three planes is called the **origin of coördinates** and is usually designated by the letter O. The lines of intersection of the planes with each other are called the **coördinate axes,** the **X-** and **Y-axes** being the intersections of the horizontal plane with the two vertical planes, and the **Z-axis** the line in which the two vertical planes meet (see Fig. 1).

It follows from elementary solid geometry that the three coördinate axes are mutually perpendicular and that each of them is perpendicular to the plane formed by the other two. In this book we shall take the positive directions on the coördinate axes

as indicated in Fig. 1.* On each of these axes a unit of measurement is adopted; we shall use equal units on the three axes.

If an arbitrary point P is now taken in three-space, we drop from it perpendiculars to the coördinate planes. Let the feet of these perpendiculars be designated by P_{xy}, P_{yz}, and P_{zx}, the subscripts indicating the planes in which these points lie (see Fig. 1). Now we lay down the following definition:

Definition I. **The *x-coördinate* of P is the measure of the line $P_{yz}P$, measured in accordance with the unit and the direction specified for the X-axis; the *y-coördinate* is the measure of the line $P_{zx}P$, measured in accordance with the unit and direction specified for the Y-axis; and the *z-coördinate* is the measure of the line $P_{xy}P$ measured in accordance with the unit and direction specified for the Z-axis.**

Notation. It will frequently be found convenient to designate the coördinates of the point P as x_P, y_P and z_P, particularly when ready identification of the points is desired. When the reference is to an arbitrary point of a specified group of points, we shall usually omit the subscript; the coördinates of a point P_1 will be denoted by x_1, y_1, z_1; those of a point P_2 by x_2, y_2, z_2; etc.

Remark 1. The coördinates of a point P are signed real numbers, the signs depending upon the position of P with respect to the coördinate planes. The x-coördinate of P is positive or negative according as P lies to the right or to the left of the YZ-plane; the y-coördinate of P is positive or negative according as P lies in front of or behind the ZX-plane; the z-coördinate is positive or negative according as P lies above or below the XY-plane.

Remark 2. It should be clear from Definition I not only that every point in three-space has a definite set of three real numbers as coördinates, but also that an arbitrary set of real numbers, taken in a definite order, determines one and only one point in three-

* The coördinate system which we have adopted is called a "left-handed system" because the thumb, first and second fingers of the left hand can be put in such a position as to suggest the positive directions along the X-, Y-, and Z-axes, particularly by a person whose finger joints have not grown stiff. If the positive direction along any one of the axes is reversed, we obtain a right-handed system. It should be clear that any two left-handed systems, and also any two right-handed systems, can be made to coincide; but that a left-handed system and a right-handed system are symmetric with respect to each other and can not be brought to coincidence if we are limited to a three-dimensional space.

space. The reader should convince himself that this point is found as the point common to three mutually perpendicular planes. The point P whose x-, y-, and z-coördinates are a, b, and c respectively will be designated by the symbol $P(a, b, c)$.

30. The Coördinate Parallelopiped of a Point. The three perpendiculars dropped from P on the coördinate planes determine, two by two, three mutually perpendicular planes. These three planes together with the coördinate planes determine a rectangular parallelopiped; we shall call this the **coördinate parallelopiped** of P, a name which we shall frequently indicate by c.p.

The coördinate axes each meet the faces of the c.p. in O and in a second point; these points are designated by P_x, P_y, P_z (see Fig. 1). The twelve edges of the c.p. are equal, four by four, to the lines whose measures are the coördinates of P. The four pairs of opposite vertices of the c.p. are O and P, P_{xy} and P_z, P_{yz} and P_x, P_{zx} and P_y; the body diagonals are the four lines which join pairs of opposite vertices. The lines PP_x, PP_y, and PP_z are perpendicular to the X-, Y-, and Z-axes respectively; the points P_x, P_y, and P_z are therefore the projections of P on the coördinate axes.

If we bear in mind the properties of the rectangular parallelopiped which are proved in elementary solid geometry, we obtain at once the following theorems.

THEOREM 1. **The x-, y-, and z-coördinates of P are equal respectively to the projections OP_x, OP_y, and OP_z of OP upon the X-, Y-, and Z-axes.**

THEOREM 2. **The square of the distance OP is equal to the sum of the squares of the coördinates of P:**

$$\overline{OP}^2 = x_P^2 + y_P^2 + z_P^2.$$

THEOREM 3. **The cosines of the angles which the line OP makes with the positive directions of the X-, Y-, and Z-axes are equal respectively to the quotients of x_P, y_P, and z_P by $\overline{OP}$.**

If we designate these angles by α_{OP}, β_{OP}, and γ_{OP} respectively, and their cosines by λ_{OP}, μ_{OP}, and ν_{OP} respectively, we have:

$$\lambda_{OP} = \cos\alpha_{OP} = \frac{x_P}{\sqrt{x_P^2+y_P^2+z_P^2}}, \quad \mu_{OP} = \cos\beta_{OP} = \frac{y_P}{\sqrt{x_P^2+y_P^2+z_P^2}},$$

$$\nu_{OP} = \cos\gamma_{OP} = \frac{z_P}{\sqrt{x_P^2+y_P^2+z_P^2}}.$$

Remark 1. The square roots in these formulas are to be taken with the positive sign; if the sign of the square root is changed, we obtain the cosines of the angles which the line OP makes with the negative directions along the coördinate axes, or, what amounts to the same, the cosines of the angles which the line PO makes with the positive directions along the axes.

Remark 2. It is important that the reader should learn to draw the coördinate parallelopipeds of points in various parts of space. The following exercises are intended chiefly to develop skill in doing this.

31. Exercises.

1. Draw the coördinate parallelopiped for each of the following points; determine their distances from the origin and the cosines of the angles which the directed lines from the origin to them make with the positive directions along the coördinate axes:

$A(-2, 5, -4)$; $B(1, -2, 3)$; $C(-1, -2, 3)$; $D(0, 4, -2)$; $E(4, 6, 7)$; $F(-4, -6, -7)$; $G(5, -2, -1)$; $H(-3, 4, 2)$; $J(6, 0, -3)$; $K(4, 3, -4)$; $L(-5, -2, -3)$; $M(-5, 3, 0)$.

2. Determine the loci of the points for which

(*a*) $x = -2$; (*b*) $y = 4$; (*c*) $z = -5$; (*d*) $x = 4$ and $y = -3$; (*e*) $y = 2$ and $z = 6$; (*f*) $z = 3$ and $x = -3$; (*g*) $x^2 + y^2 + z^2 = 9$.

3. Determine the coördinates of the points which are symmetric with $P(a, b, c)$ with respect to

(*a*) the XZ-plane; (*b*) the Y-axis; (*c*) the origin; (*d*) the Z-axis; (*e*) the YZ-plane; (*f*) the X-axis; (*g*) the XY-plane.

4. Develop one or more algebraic conditions which are satisfied by the coördinates of the points which lie

(*a*) on a sphere of radius 4 which has its center at the origin;
(*b*) on a plane which cuts the Y-axis perpendicularly at a point 5 units behind the origin;
(*c*) on a line parallel to the Z-axis and through the point $A(3, -4, 1)$;
(*d*) on a plane parallel to the YZ-plane and passing through the point $A(-1, 2, 1)$;
(*e*) on a circle in the ZX-plane whose center is at the point $C(0, 0, 4)$ and whose radius is 3;
(*f*) on a line perpendicular to the XZ-plane and through the point $A(-2, -3, -1)$;
(*g*) on the line determined by the origin and the point $A(2, -1, 1)$.

5. The point P lies on a line through the origin which makes with the positive directions on the coördinate axes angles whose cosines are equal to $\frac{1}{2}$, $-\frac{1}{2}$, and $\frac{\sqrt{2}}{2}$, and the distance OP is equal to 3. Determine the coördinates of P.

6. Prove that the line from 0 to $A(1, 1, 1)$ makes equal angles with the positive directions on the three coördinate axes.

7. Prove that the lines which join the origin to $P(a, b, c)$ and to $Q(ka, kb, kc)$ make equal or supplementary angles with the positive directions on the coördinate axes.

8. Prove that the sum of the squares of the cosines of the angles which the line OP makes with the positive directions on the coördinate axes is equal to 1.

32. Two Points. The c.p. of the point P is a rectangular parallelopiped whose faces are parallel to, or lie in, the coördinate planes and of which the origin and P are opposite vertices, while the other vertices all lie in the coördinate planes. We construct now a rectangular parallelopiped whose faces are parallel to, or lie in, the coördinate planes, but of which two opposite vertices are to be two arbitrarily assigned points P and Q. The construction, of which the result is shown in Fig. 2, is carried out most readily as follows:

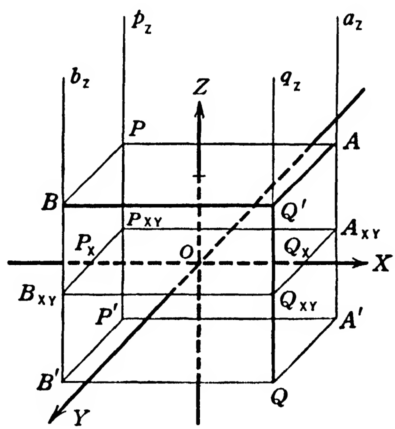

Fig. 2

Through the points P_{xy} and Q_{xy} we draw lines parallel to the X- and Y-axes, so as to form the rectangle $P_{xy}A_{xy}Q_{xy}B_{xy}$; through the four vertices of this rectangle we draw lines p_z, a_z, q_z, and b_z parallel to the Z-axis which are the vertical boundaries of the side faces of the parallelopiped. To complete the construction, we draw through P a line parallel to the X-axis, meeting a_z in A; through A a line parallel to the Y-axis, meeting q_z in Q'; through Q' a line parallel to the X-axis, meeting b_z in B; and through B

a line parallel to the Y-axis, meeting p_z in P. Starting from Q, we locate in a similar manner the vertices B', P', and A' of the parallelopiped.

Remark 1. The parallelopiped whose construction is described above will be called the coördinate parallelopiped (c.p.) of P and Q. It is important that the reader develop skill in carrying out this construction; a number of valuable results can be obtained readily by means of it.

Remark 2. The construction of the c.p. of two points P and Q can be carried out equally well if we start with the points P_{yz} and Q_{yz}, or with the points P_{zx} and Q_{zx}.

Since the line $P'A'$ is parallel to the X-axis, the segment $\overline{P'A'}$ is equal to the segment $\overline{P_xQ_x}$ of the X-axis determined by the projections of P and Q on the X-axis. Since O, P_x, and Q_x are points on the same directed line, namely, the X-axis, we know moreover that

$$OP_x + P_xQ_x + Q_xO = 0$$

that is,

$$x_P + P_xQ_x - x_Q = 0,$$

in virtue of Theorem 1, Section 30, page 51. Hence we conclude that

$$\text{Proj}_X PQ = P_xQ_x = x_Q - x_P.$$

Leaving the proof of similar formulas for the projections of the segment PQ on the Y- and Z-axes to the reader, we state the following theorem.

THEOREM 4. **The projections on the coördinate axes of a directed segment of a straight line are equal to the differences between the corresponding coördinates of the end point and those of the initial point of the segment.**

We observe now that the twelve edges of the c.p. of P and Q have lengths equal, four by four, to the numerical value of the differences between the coördinates of P and Q. If we make use once more of the property of the body diagonal of a rectangular parallelopiped which was brought forward in Section 30, we obtain the following extension of Theorem 2.

THEOREM 5. **The square of the distance PQ is equal to the sum of the squares of the differences of the coördinates of P and of Q, that is,**

$$\overline{PQ}^2 = (x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2.$$

33. Direction Cosines of a Line. We recall that the angle between two lines l and m which do not lie in the same plane is defined as the angle between any two concurrent lines of which one is parallel to l and the other to m. This extension of the concept "angle between two lines" enables us to speak of the angles which an arbitrary line makes with the coördinate axes and gives significance therefore to the following definition.

DEFINITION II. **The *direction angles* of a directed line are the angles between $-180°$ and $+180°$ which the directed line makes with the positive directions of the coördinate axes. The *direction cosines* of a directed line are the cosines of its direction angles.**

Notation. Whenever it is desirable to specify the directed line to which reference is made, the direction angles of the line PQ will be designated by the symbols α_{PQ}, β_{PQ}, and γ_{PQ}; its direction cosines by λ_{PQ}, μ_{PQ}, and ν_{PQ}. Similar notations will be used for the direction angles and the direction cosines of an undirected line l. When it is not essential to specify the line, the subscripts will be omitted.

On the basis of this definition we obtain, from a consideration of the c.p. of P and Q and by using some of the properties of the rectangular parallelopiped mentioned in the second paragraph of Section 30 (page 51), the following extension of Theorem 3.

THEOREM 6. **The direction cosines of the directed line PQ are equal to the quotients of the differences between the coördinates of Q and those of P by the distance $\overline{PQ}$; that is,**

$$\lambda_{PQ} = \cos\alpha_{PQ} = \frac{x_Q - x_P}{\sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}},$$

$$\mu_{PQ} = \cos\beta_{PQ} = \frac{y_Q - y_P}{\sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}},$$

$$\nu_{PQ} = \cos\gamma_{PQ} = \frac{z_Q - z_P}{\sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}}.$$

Remark. The square root here, as in Theorem 3, is to be taken with the positive sign; a change of sign in the square root leads to the direction cosines of the line QP. In the case of an undirected line l, the sign of the square root remains ambiguous, the two signs corresponding to the two directions which may be specified

upon the line. The formulas of Theorem 6 lead to the following very useful corollaries.

COROLLARY 1. **The direction cosines of an undirected line are proportional to the differences between the coördinates of any two of its points.**

COROLLARY 2. **The direction cosines of an undirected line through the origin are proportional to the coördinates of any one of its points.**

COROLLARY 3. **The coördinates of any point on the line which joins the origin to a point P are proportional to the coördinates of P.**

(Compare Exercise 7, Section 31, p. 53.)

Furthermore we obtain the following important result.

THEOREM 7. **The sum of the squares of the direction cosines of a line is equal to 1.**

Proof. If we choose any two points P and Q on the line and express the direction cosines of the line in terms of the coördinates of P and of Q, as in Theorem 6, we find that

$$\lambda^2 + \mu^2 + \nu^2 = \frac{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2} = 1.$$

Remark. The formula proved in Theorem 7 will be used repeatedly in the sequel. We shall use the phrase "admissible values of λ, μ, ν" to indicate values of these variables for which $\lambda^2 + \mu^2 + \nu^2 = 1$. Its principal use will be to enable us to determine the direction cosines of a line if we merely know numbers to which they are proportional. This will in most cases relieve us of the necessity of actually determining the direction cosines and make it possible to operate with their ratios. A special case of Theorem 7 is contained in Exercise 8, Section 31, (page 53).

THEOREM 8. **If the direction cosines of an undirected line are proportional to three given numbers, then their actual values are equal to these numbers, each divided by the square root of the sum of their squares.**

For, if $\lambda = ka$, $\mu = kb$ and $\nu = kc$, then it follows from Theorem 7 that $k^2(a^2 + b^2 + c^2) = 1$, so that $k = \dfrac{1}{\sqrt{a^2 + b^2 + c^2}}\cdot$ The ambiguity of sign in the square root corresponds to the possibility of two directions on the undirected line.

34. Three Collinear Points. If A, B and P are points on a line, the direction cosines of the segments AP and AB are either equal or else equal numerically but opposite in sign, thus:

$$\lambda_{AP} = \pm\lambda_{AB}, \quad \mu_{AP} = \pm\mu_{AB}, \quad \nu_{AP} = \pm\nu_{AB},$$

where the upper signs are to be used if the segments AP and AB have the same direction and the lower signs if they have opposite directions. And, if the points A, B and P are not collinear, not all of these relations can hold. By means of Theorem 6 we derive from these relations the following equations:

$$\frac{x_P - x_A}{\overline{AP}} = \pm\frac{x_B - x_A}{\overline{AB}}, \quad \frac{y_P - y_A}{\overline{AP}} = \pm\frac{y_B - y_A}{\overline{AB}},$$

$$\frac{z_P - z_A}{\overline{AP}} = \pm\frac{z_B - z_A}{\overline{AB}};$$

or

$$\frac{x_P - x_A}{x_B - x_A} = \frac{y_P - y_A}{y_B - y_A} = \frac{z_P - z_A}{z_B - z_A} = \pm\frac{\overline{AP}}{\overline{AB}}.$$

Moreover, if the segments AP and AB have the same direction we can take $AP = +\overline{AP}$ and $AB = +\overline{AB}$; whereas if they have opposite directions, we can take $AP = +\overline{AP}$ and $AB = -\overline{AB}$. It follows therefore that we have $\dfrac{AP}{AB} = +\dfrac{\overline{AP}}{\overline{AB}}$ or $-\dfrac{\overline{AP}}{\overline{AB}}$ according as the segments AP and AB have the same or opposite directions. We obtain therefore the following important result.

THEOREM 9. **The necessary and sufficient condition that the point $P(x, y, z)$ shall lie on the line through A and B is that the coördinates x, y, z must satisfy the linear equations**

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A} = \frac{AP}{AB}.$$

Remark 1. It should be clear that by starting with the segments BP and BA we find the equations

$$\frac{x - x_B}{x_A - x_B} = \frac{y - y_B}{y_A - y_B} = \frac{z - z_B}{z_A - z_B} = \frac{BP}{BA};$$

and if we use the segments AP and PB, the resulting equations are

$$\frac{x - x_A}{x_B - x} = \frac{y - y_A}{y_B - y} = \frac{z - z_A}{z_B - z} = \frac{AP}{PB}.$$

Remark 2. The formulas established in Theorem 9, and also those given in Remark 1, enable us to determine the coördinates of the point P on the line AB as soon as the coördinates of the points A and B are known and also the ratio $\frac{AP}{PB}$ of the segments AP and PB in which P divides the segment AB. For, if $AP : PB = r_1 : r_2$, then $AP : AB = r_1 : r_1 + r_2$, and $BP : BA = r_2 : r_1 + r_2$. Hence we have

$$\frac{x - x_A}{x_B - x} = \frac{r_1}{r_2}, \quad \frac{x - x_A}{x_B - x_A} = \frac{r_1}{r_1 + r_2}, \quad \frac{x - x_B}{x_A - x_B} = \frac{r_2}{r_1 + r_2}.$$

From either of these equations we find for the x-coördinate of P:

$$x_P = \frac{r_2 x_A + r_1 x_B}{r_1 + r_2};$$

and similar results are obtained for y_P and z_P. We have therefore the following further result.

Corollary. **If the points A, B, and P are in a straight line and if $AP : PB = r_1 : r_2$, then the coördinates x, y, z of P are given by the formulas:**

$$x = \frac{r_2 x_A + r_1 x_B}{r_1 + r_2}, \quad y = \frac{r_2 y_A + r_1 y_B}{r_1 + r_2}, \quad z = \frac{r_2 z_A + r_1 z_B}{r_1 + r_2}.$$

Remark 3. The formulas of this corollary are very useful for later developments. But they hide to some extent the simple geometrical fact from which they have been derived and which finds more direct expression in the formulas of Theorem 9 and Remark 1. It is advisable therefore, particularly in the beginning and in numerical problems, to go back to these earlier formulas rather than merely to substitute in the formulas of the corollary.

Remark 4. The first three terms in the equations of Theorem 9 give two independent linear equations which the coördinates of any point on AB must satisfy; the same statement may be made for the equations in Remark 1. If we consider also the last term in each of these cases — and here the last set of equations in Remark 1 is particularly useful — they give us three equations which express the coördinates of an arbitrary point P on the line in terms of the coördinates of A and B, and of one parameter, or auxiliary variable, namely, $r = \frac{r_1}{r_2}$, that is, in terms of the ratio r

of the segments into which P divides the segment AB. This parameter r varies as the point P moves along the line AB. When P coincides with A, $r = 0$; as P moves from A to B r increases. When P lies outside the segment AB, either on the side of A or on the side of B, r is negative; and as P moves off indefinitely along the line in either direction, r tends towards -1. For

$$r = \frac{AP}{PB} = \frac{AB + BP}{PB} = \frac{AB}{PB} - 1;$$

since AB is fixed and PB increases numerically when P moves off along the line, the first term on the right tends toward zero and hence r tends toward -1. The reader will find it worth while to make clear to himself in detail the manner in which the parameter r varies as P occupies various positions on the line AB.

If we combine Theorems 6 and 9, we are led to the following valuable theorem.

THEOREM 10. **The necessary and sufficient condition that a point $P(x, y, z)$ shall lie on the directed line through A whose direction cosines are λ, μ, and ν is that the coördinates x, y, and z must satisfy the equations**

$$\frac{x - x_A}{\lambda} = \frac{y - y_A}{\mu} = \frac{z - z_A}{\nu} = AP.$$

For if P lies on the specified line and if B is another arbitrary, but fixed, point on the line, then the equations of Theorem 9 hold. But, in accordance with Theorem 6, we have

$$x_B - x_A = \lambda \cdot AB, \quad y_B - y_A = \mu \cdot AB, \quad z_B - z_A = \nu \cdot AB.$$

If these values are substituted in the equations of Theorem 9, the desired result is obtained.

COROLLARY 1. **The necessary and sufficient condition that a point $P(x, y, z)$ shall lie on the undirected line through the point A whose direction cosines are proportional to l, m and n is that the coördinates x, y and z shall satisfy the equations**

$$\frac{x - x_A}{l} = \frac{y - y_A}{m} = \frac{z - z_A}{n} = \frac{AP}{\sqrt{l^2 + m^2 + n^2}}.$$

This corollary follows from Theorem 10 in combination with Theorem 8. The ambiguity in the sign of the square root corresponds to the possibility of two directions on the line of which only the ratios of the direction cosines are given.

The observation made in Remark 4 (page 58) leads from Theorem 10 and Corollary 1 to two further results of importance.

Corollary 2. **The coördinates of a point $P(x, y, z)$ on the line through A whose direction cosines are equal to λ, μ, ν are given by the equations**

$$x = x_A + \lambda s, \quad y = y_A + \mu s, \quad z = z_A + \nu s,$$

where s designates the signed measure of the directed segment AP; conversely, any point P whose coördinates are equal to these expressions lies on the specified line at a directed distance from A equal to s.

Corollary 3. **The coördinates of a point $P(x, y, z)$ on the undirected line through A whose direction cosines are proportional to l, m, n are given by the equations**

$$x = x_A + lt, \quad y = y_A + mt, \quad z = z_A + nt, \quad \text{where } t = \frac{AP}{\sqrt{l^2 + m^2 + n^2}};$$

conversely, any point P whose coördinates are given by these expressions lies on the specified line at a distance from A equal to $t\sqrt{l^2 + m^2 + n^2}$.

Remark. The equations of Corollary 1 are frequently referred to as the "symmetric equations of the line"; those of Corollaries 2 and 3 as the "parametric equations of the line." The variable s, or t, which changes as P moves along the line, is the parameter. The terminology "equations of a line" will be more fully justified in Chapter IV (see Section 47, page 83).

The parametric equations of the line are used a great deal in the sequel. The reader is urged to master thoroughly the methods by which they have been obtained. It should be observed moreover that the equations stated in the Corollary of Theorem 9 are also parametric equations of the line, the parameter being the ratio r (see Remark 4, page 58).

35. Exercises.

1. Construct the coördinate parallelopipeds of the following pairs of points, and determine their distances and the direction cosines of the lines joining them:

(*a*) $A(5, 2, -1)$ and $B(-3, -4, 2)$; (*b*) $A(2, 4, 5)$ and $B(7, 1, 1)$; (*c*) $A(2, 3, 4)$ and $B(5, -2, 7)$; (*d*) $A(3, -2, -1)$ and $B(-3, 4, 5)$; (*e*) $A(-4, 3, 5)$ and $B(-4, -2, 0)$; (*f*) $A(3, 4, 5)$ and $B(-3, -4, -5)$; (*g*) $A(0, 3, 6)$ and $B(4, -1, 6)$; (*h*) $A(-2, 3, 5)$ and $B(-2, 3, -1)$.

2. Determine which of the following sets of points are collinear:

(a) $A(3, -1, 4)$, $B(-2, 4, -1)$, $C(1, 1, 2)$; (b) $A(0, 0, 0)$, $B(2, 5, -3)$, $C(4, 10, -6)$; (c) $A(1, -2, 3)$, $B(-1, 2, -3)$, $C(-3, 5, 0)$; (d) $A(-2, 2, 3)$, $B(1, -1, 0)$, $C(7, -7, -3)$; (e) $A(0, -3, 1)$, $B(4, -2, -1)$, $C(2, -4, 3)$; (f) $A(5, 2, 7)$, $B(1, 5, 5)$, $C(-3, 8, 3)$.

3. Determine the coördinates of the point at which the segment from $A(-3, 2, 5)$ to $B(5, -4, 2)$ is bisected; also the coördinates of the points at which this segment is trisected.

4. The line of the preceding problem is extended beyond B to a point C such that (a) $BC = AB$; (b) $BC = 3\,AB$; (c) $BC = \frac{1}{2}\,AB$. Determine the coördinates of C in each case.

5. Find the coördinates of the center of mass of the homogeneous triangle whose vertices are $A(-2, 5, 4)$, $B(3, -1, -2)$ and $C(8, -7, 4)$. (The center of mass of a homogeneous triangle is the point of intersection of the medians.)

6. Determine the coördinates of the point in which the side AC of the triangle of Exercise 5 is met by the bisectors of the interior and the exterior angles at B. Find also the direction cosines of the bisectors.

7. Show that the points $A(-3, 2, 5)$, $B(1, 0, 1)$ and $C(11, -5, -9)$ are collinear and determine the ratio of the segments $AC : CB$.

8. On a line through the point $A(5, -4, 2)$ whose direction cosines are proportional to $2, -1$ and -2, a point B is determined such that $AB = 4$. Find the coördinates of B. How many positions are possible for B?

9. Determine the center of mass of the homogeneous triangle whose vertices are at $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3$.

10. Show that the three lines which join the midpoints of the three pairs of opposite edges of the tetrahedron $P_1P_2P_3P_4$ have a common midpoint. This common midpoint is called the center of mass of the homogeneous tetrahedron $P_1P_2P_3P_4$. (Opposite edges of a tetrahedron are edges which have no point in common.)

11. Prove that any vertex of a homogeneous tetrahedron, the center of mass of the opposite face and the center of mass of the tetrahedron are collinear; determine the ratio of the segments in which the center of mass of the tetrahedron divides the segment determined by the other two points.

12. Show that if and only if $P(x, y, z)$ lies on the sphere of radius r, whose center is at $C(a, b, c)$, the coördinates x, y, z satisfy the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

13. Determine the equation satisfied by the coördinates of all points which are at the same distance from $A(-2, 1, -3)$ as from $B(4, 2, 0)$.

14. Determine the equation satisfied by the coördinates of all points on the surface of the sphere whose center is at $C(3, -2, -3)$ and whose radius is 5.

15. Determine the equation satisfied by the coördinates of all points whose distance from $A(1, -3, 4)$ is twice as great as their distance from $B(-2, 0, 2)$.

36. The Angle Between Two Lines. The Projection Method. To calculate the angle between two lines whose direction cosines

are known, we shall use a method which will find frequent application in the sequel and which is based on the following two theorems.

THEOREM 11. **The projection of a segment AB of a directed line l upon a directed line m is equal to the product of AB by the cosine of the angle between the two directed lines.**

Proof. We distinguish two cases, according as the lines l and m do or do not lie in one plane.

(a) For the case when the lines l and m lie in one plane, the proof can be found in most books on Plane Analytical Geometry and in some books on Trigonometry.* For this reason we shall not repeat the proof here.

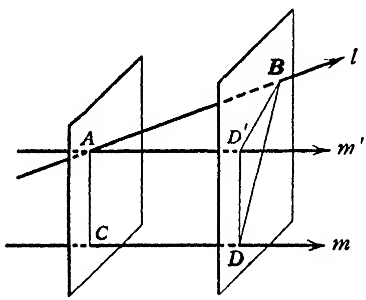

FIG. 3

(b) If the lines l and m are not coplanar, let the angle between them be θ. We construct a line m' through the point A on l, parallel to m (see Fig. 3); the angle between m' and l will then also be equal to θ. Through A and B we construct planes perpendicular to m; these planes will then also be perpendicular to m'. If the points in which these planes meet m and m' are C, D and A, D' respectively, then $CD = AD'$ (Why?). From these facts, in combination with part (a) of this proof, we conclude that

$$\mathrm{Proj}_m AB = CD = AD' = AB \cos \theta.$$

This proves our theorem.

THEOREM 12. **The sum of the projections upon a directed line m of the segments of a closed broken line in space is equal to zero.**

Proof. If the vertices of the broken line in space are A, B, C, . . . , P and if their projections on the directed line m are A', B', C', . . . , P', we have to show that $A'B' + B'C' + \ldots + P'A' = 0$. That this is indeed the case follows from a fundamental theorem, of which a proof is found in books on Plane Analytical Geometry and on Trigonometry,† according to which the sum of

* See, for example, the author's Plane Trigonometry, p. 36.

† See, for example, the author's Plane Trigonometry, p. 4.

the directed segments of a line, of which the end point of the last segment coincides with the initial point of the first segment, is equal to zero. We conclude therefore that

$$\text{Proj}_m AB + \text{Proj}_m BC + \ldots + \text{Proj}_m PA = A'B' + B'C' + \ldots + P'A' = 0.$$

We proceed now to the determination of the angle θ between two directed lines l and m. We construct first the c.p. of two points A and B selected arbitrarily on one of the lines (see Fig. 4 in which A and B are taken on l); and we apply Theorem 12 to the closed broken line $ABDCA$. Thus we obtain the equation

$$\text{Proj}_m AB + \text{Proj}_m BD + \text{Proj}_m DC + \text{Proj}_m CA = 0.$$

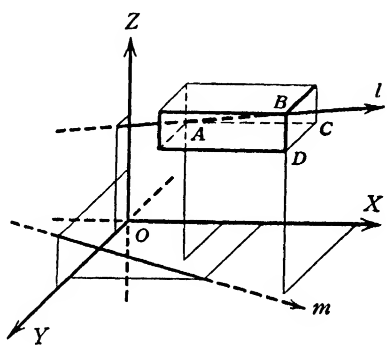

FIG. 4

The segment AB lies on l, the segments BD, DC and CA on lines which are parallel to the Z-, Y-, and X-axes; the angles which these lines make with the line m are respectively equal to θ, γ_m, β_m, and α_m, the last three angles being the direction angles of m. The last equation written above leads therefore by means of Theorem 11 to the statement that

$$AB \cos\theta + BD\,\nu_m + DC\,\mu_m + CA\,\lambda_m = 0.$$

Moreover BD, DC and CA are equal to the projections of BA on the Z-, Y-, and X-axes respectively, so that it follows from Theorem 11 that

$$BD = BA\,\nu_l, \quad DC = BA\,\mu_l \quad \text{and} \quad CA = BA\,\lambda_l.$$

If we substitute these values in the preceding equation and remember that $BA = -AB \neq 0$, we obtain the following result:

$$\cos\theta = \lambda_l\lambda_m + \mu_l\mu_m + \nu_l\nu_m.$$

THEOREM 13. **The cosine of the angle between two directed lines is equal to the sum of the products of their corresponding direction cosines.**

Remark. If we restrict angles between directed lines to lie between $-180°$ and $+180°$, the result contained in Theorem 13 determines the magnitude of angle θ but leaves the sign of this angle ambiguous. This ambiguity corresponds to the fact that either of the two lines may be taken as the initial side of the angle.

COROLLARY 1. **The necessary and sufficient condition that the lines l and m are perpendicular is that $\lambda_l\lambda_m + \mu_l\mu_m + \nu_l\nu_m = 0$.**

COROLLARY 2. **If the direction cosines of two undirected lines are proportional to l_1, m_1, n_1 and l_2, m_2, n_2, the necessary and sufficient condition for the perpendicularity of the lines is that $l_1l_2 + m_1m_2 + n_1n_2 = 0$.**

By means of Theorem 13 we can determine the numerical values of all the trigonometric ratios of the angles between two directed lines. On account of its special interest we develop a formula for the sine of these angles. This is done most conveniently by means of the following auxiliary theorem, which is of some interest on its own account.

LEMMA. **The following identity holds between any six numbers, real or complex, a, b, c and a_1, b_1, c_1:**

$$(a^2 + b^2 + c^2)(a_1^2 + b_1^2 + c_1^2) - (aa_1 + bb_1 + cc_1)^2$$
$$= \begin{vmatrix} b & c \\ b_1 & c_1 \end{vmatrix}^2 + \begin{vmatrix} c & a \\ c_1 & a_1 \end{vmatrix}^2 + \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}^2.$$

The proof of this formula requires merely that the indicated operations on both sides of the equation shall be carried out; the identity of the two sides will then at once become apparent.

If this lemma is applied to the direction cosines λ_l, μ_l, ν_l and λ_m, μ_m, ν_m of the lines l and m, we find that

$$(\lambda_l^2 + \mu_l^2 + \nu_l^2)(\lambda_m^2 + \mu_m^2 + \nu_m^2) - (\lambda_l\lambda_m + \mu_l\mu_m + \nu_l\nu_m)^2 =$$
$$\begin{vmatrix} \mu_l & \nu_l \\ \mu_m & \nu_m \end{vmatrix}^2 + \begin{vmatrix} \nu_l & \lambda_l \\ \nu_m & \lambda_m \end{vmatrix}^2 + \begin{vmatrix} \lambda_l & \mu_l \\ \lambda_m & \mu_m \end{vmatrix}^2.$$

But in view of Theorems 7 and 13 this result leads to the equation

$$1 - \cos^2\theta = \begin{vmatrix} \mu_l & \nu_l \\ \mu_m & \nu_m \end{vmatrix}^2 + \begin{vmatrix} \nu_l & \lambda_l \\ \nu_m & \lambda_m \end{vmatrix}^2 + \begin{vmatrix} \lambda_l & \mu_l \\ \lambda_m & \mu_m \end{vmatrix}^2.$$

THEOREM 14. **The square of the sine of the angle between two directed lines is equal to the sum of the squares of the two-rowed minors which can be formed from the matrix constituted by the two sets of**

direction cosines of the lines; that is,

$$\sin^2\theta = \begin{vmatrix} \mu_l & \nu_l \\ \mu_m & \nu_m \end{vmatrix}^2 + \begin{vmatrix} \nu_l & \lambda_l \\ \nu_m & \lambda_m \end{vmatrix}^2 + \begin{vmatrix} \lambda_l & \mu_l \\ \lambda_m & \mu_m \end{vmatrix}^2.$$

37. Exercises.

1. Determine the cosines of the angles between the lines whose direction cosines λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2 are given by the following data:

$$\begin{array}{llll} (a) & \lambda_1 : \mu_1 : \nu_1 = -2 : 1 : 2, & & \lambda_2 : \mu_2 : \nu_2 = 2 : 6 : -3 \\ (b) & \lambda_1 : \mu_1 : \nu_1 = 2 : 3 : 6, & & \lambda_2 : \mu_2 : \nu_2 = 3 : 14 : 18 \\ (c) & \lambda_1 : \mu_1 : \nu_1 = 3 : -2 : -6, & & \lambda_2 : \mu_2 : \nu_2 = 1 : -2 : -2 \\ (d) & \lambda_1 : \mu_1 : \nu_1 = 4 : 0 : 3, & & \lambda_2 : \mu_2 : \nu_2 = 1 : 2 : 3 \end{array}$$

2. Determine the cosines of the angles formed by the sides of the triangle whose vertices are the points $A(3, -1, 4)$, $B(-2, 4, -1)$ and $C(1, 1, -2)$.

3. Test for perpendicularity the pairs of lines whose direction cosines are given by the following data:

$$\begin{array}{lll} (a) & \lambda_1 : \mu_1 : \nu_1 = -2 : 1 : 2, & \lambda_2 : \mu_2 : \nu_2 = 1 : -2 : 2 \\ (b) & \lambda_1 : \mu_1 : \nu_1 = 3 : -1 : 2, & \lambda_2 : \mu_2 : \nu_2 = 1 : 1 : -1 \\ (c) & \lambda_1 : \mu_1 : \nu_1 = 2 : -3 : 6, & \lambda_2 : \mu_2 : \nu_2 = 1 : 2 : 0 \end{array}$$

4. Develop a formula for the tangent, the cotangent, the secant and the cosecant of the angle between two lines.

5. Determine the direction cosines of a line which is perpendicular to the two lines whose direction cosines are proportional to $3 : -2 : 4$ and $1 : 3 : -2$.

6. Determine the direction cosines of a line which makes equal angles with the three lines whose direction cosines are proportional to 1, 4, 8; to 8, -1, 4; and to 1, 2, -2.

38. Miscellaneous Exercises.

1. Determine the conditions which the coördinates of a point must satisfy in order to be equally distant from the points $A(2, -1, 0)$, $B(-3, 2, 1)$ and $C(1, 3, -2)$.

2. Solve the same problem for the points $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3$.

3. Determine a point which is equally distant from the four points $A(2, 1, 3)$, $B(1, 1, 2)$, $C(2, 0, 5)$, and $D(2, 0, 3)$.

4. Establish the condition on the coördinates of the four points $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3, 4$, necessary and sufficient for the existence of a single point that is equally distant from these four points.

5. Set up the equation which is satisfied by the coördinates of any point on the sphere which passes through the four points $A(3, -2, 4)$, $B(2, -3, 2)$, $C(4, -2, 2)$ and $D(5, -1, 3)$.

6. Determine the direction cosines of a line perpendicular to the two lines whose direction cosines are λ_i, μ_i, ν_i, $i = 1, 2$. Does this problem always have a solution? Does it ever have more than one solution?

7. Remembering that a line is perpendicular to a plane if it is perpendicular to two lines in the plane, determine the direction cosines of a line which is

perpendicular to the plane of the triangle whose vertices are the points $A(3, -2, 4)$, $B(4, 0, 2)$ and $C(0, -4, -2)$.

8. Determine the condition which must be satisfied by the direction cosines λ_i, μ_i, ν_i, $i = 1, 2, 3$ of three lines in order that there may exist one or more lines perpendicular to these lines.

9. Show that if the coördinates of the three points $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3$ satisfy an equation of the form $ax + by + cz = 0$, in which a, b, c are not all zero, then there exists at least one line which is perpendicular to the three lines OP_i, $i = 1, 2, 3$.

10. Determine the direction cosines of a line which is to make equal angles with the three lines whose direction cosines are λ_i, μ_i, ν_i, $i = 1, 2, 3$. Does this problem always have a solution? Can it have more than one solution?

11. Determine the direction cosines of a line which makes equal angles with the three lines connecting the origin with the points $A(-2, 3, 6)$, $B(1, -8, 4)$, and $C(1, 2, -2)$.

12. The line which joins $A(1, 3, 5)$ to $B(15, -15, 8)$ is produced beyond B to a point P such that $BP = 5$. Determine the coördinates of P.

13. Prove that every point whose coördinates satisfy the equation

$$x^2 + y^2 + z^2 + 2\,ax + 2\,by + 2\,cz + d = 0$$

lies on the surface of a sphere. Determine the center and the radius of this sphere.

14. Determine the center and the radius of each of the spheres represented by the following equations:

(a) $x^2 + y^2 + z^2 - 4\,x + 6\,y - 6\,z - 3 = 0$
(b) $x^2 + y^2 + z^2 + 6\,x - 2\,y + 4\,z - 2 = 0$
(c) $x^2 + y^2 + z^2 + 2\,x - 6\,y + 8\,z + 26 = 0$
(d) $x^2 + y^2 + z^2 - 8\,x + 4\,y + 6\,z - 33 = 0$

15. Two lines m_1 and m_2 meet at a point O under an angle $\theta, \neq 180°$. A line l, not necessarily in the same plane with m_1 and m_2, makes angles α_1 and α_2 with m_1 and m_2 respectively and an angle β with a line in the plane of m_1 and m_2, perpendicular to m_1, and on the same side of m_1 as m_2. Prove that $\cos\beta \sin\theta = \cos\alpha_2 - \cos\alpha_1 \cos\theta = \begin{vmatrix} 1 & \cos\theta \\ \cos\alpha_1 & \cos\alpha_2 \end{vmatrix}$. *Hint:* Take O as the origin of a frame of reference, the line m_1 as X-axis, and the pl[illegible]and m_2 as XY-plane.

16. If, in the configuration of the preceding exercise, γ is [illegible]ngle which l makes with a line perpendicular to the plane of m_1 and m_2, then $\cos\gamma \sin\theta = \pm [1 - \cos^2\alpha_1 - \cos^2\alpha_2 - \cos^2\theta + 2\cos\alpha_1\cos\alpha_2\cos\theta]^{\frac{1}{2}}$

$$= \pm \begin{vmatrix} 1 & \cos\theta & \cos\alpha_1 \\ \cos\theta & 1 & \cos\alpha_2 \\ \cos\alpha_1 & \cos\alpha_2 & 1 \end{vmatrix}^{\frac{1}{2}};$$

the $+$ or $-$ sign is to be used according as l and the perpendicular to the plane of m_1 and m_2 point to the same side or to opposite sides of this plane.

CHAPTER IV

PLANES AND LINES

39. Surfaces and Curves. If a point P is to be chosen at random in space, its determination will, in the coördinate system which we have used thus far, depend upon three independent choices of arbitrary real numbers, namely, of an x-, a y-, and a z-coördinate, each of which can be selected without consideration of the selections made for the other two. This fact is expressed in the statement that **a point in space has three degrees of freedom.**

If a point is to be chosen at random on a surface such as we are likely to meet in ordinary experience (we may think here of surfaces which limit the objects in our environment, such as tables, bottles, lamp shades, trees, etc.) the determination will depend upon two independent choices of arbitrary real numbers; for, after two coördinates have been chosen at random, the x- and y-coördinates for example, the third one, the z-coördinate, must be so chosen as to furnish a point on the surface (see Fig. 5); and this will leave, at least in the case of surfaces of ordinary experience, in general a choice among a finite number of values. For this reason **a poi[illegible]rface is said to have two degrees of freedom.**

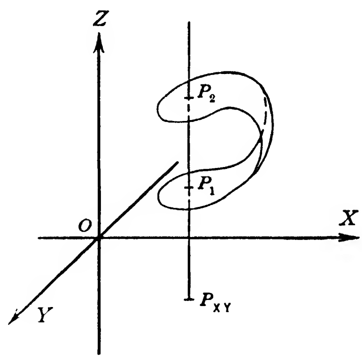

Fig. 5

And if [illegible] is to be chosen at random on a curve (again our reference is in the first place to curves of common experience) only one coördinate can be selected arbitrarily; **a point on a curve is therefore said to have one degree of freedom.**

It should be clear from this discussion that to restrict a point in three-space to a surface, we have to impose one condition on its coördinates; and to limit a point in three-space to a curve, we shall have to subject its coördinates to two independent conditions.

These considerations, vague and inconclusive though they are, suffice perhaps to indicate that there is some justification for the following definitions.

Definition I. **A surface is the totality of all points in three-space whose Cartesian coördinates satisfy *one* equation.**

Remark. The equation which is thus laid down by definition as the algebraic counterpart of the surface, is called the **equation of the surface**; and the surface is referred to as the **locus of the equation.**

Definition II. **A curve is the totality of all those points in three-space whose Cartesian coördinates satisfy *two* independent equations.**

We shall speak of the "equations of a curve" and of the "locus of a pair of equations."

Remark. These definitions do not specify sharply the concepts "surface" and "curve" because the word "equation" used in them is left without specification. According as the class of equations that is taken into consideration is widened or narrowed, we shall presumably enlarge or restrict the concepts "surface" and "curve." The remarks preceding the definitions are intended to make clear that the surfaces and curves of ordinary experience are included among the "surfaces" and "curves" introduced by the definitions.

The surfaces and curves which are the loci of algebraic equations and pairs of algebraic equations, respectively, are called "algebraic surfaces" and "algebraic curves." In this book we shall be concerned exclusively with surfaces (and curves) which are loci of the simplest types of algebraic equations in three variables (pairs of such equations), namely, of algebraic equations of the first and second degrees. We shall, however, have to deal occasionally with loci of equations of a more general character; and we shall begin with some considerations of a general nature.

40. Cylindrical Surfaces. Systems of Planes. What can be said about the space locus of an equation like $x^2 + y^2 = 4$, in which only two of the variables are present? The plane locus of such an equation consists of the points on the XY-plane, whose coördinates satisfy the given equation; it is therefore a plane curve. Since the equation does not restrict the z-coördinate, a

point $P(a, b, c)$ will belong to its locus if and only if the point $P_{xy}(a, b, 0)$ does, that is, if and only if the projection of P on the XY-plane belongs to the plane locus of the equation. Consequently the space locus of this equation will be generated by a line which moves parallel to the Z-axis and which passes successively through the points of the plane locus of the equation.

We introduce now the following terminology:

DEFINITION III. **A *cylindrical surface* is a surface generated by a line which moves in such a way as to be always parallel to a fixed line and in such a way as to pass through the points on a fixed plane curve. Any position of the moving line is called a *generating line* (*generatrix*); the fixed plane curve is called the *directrix*. If the generatrix is perpendicular to the plane of the directrix, we have a *right cylindrical surface*; if not, an *oblique cylindrical surface*.**

By the aid of this terminology we can express the result of the foregoing discussion in the following form.

THEOREM 1. **The locus of the equation $f(x, y) = 0$ is a right cylindrical surface whose generating line is parallel to the Z-axis and whose directrix is the plane locus of the equation.**

Remark 1. Similar theorems hold of course concerning the loci of equations from which the variable x or the variable y is absent. It will be good practice for the reader to present in full the argument for these cases. The locus of the particular equation $x^2 + y^2 = 4$ is a right circular cylindrical surface, whose generating line is parallel to the Z-axis and whose directrix is a circle in the XY-plane with center at the origin and radius equal to 2.

Remark 2. By means of this theorem the whole field of Plane Analytical Geometry is seen to be a province in the domain of Solid Analytical Geometry. For it shows that the determination of the space loci of equations in two Cartesian variables depends upon the determination of the plane loci of such equations.

In particular the space locus of a linear equation in two variables, for example, of the equation $ay + bz + c = 0$, is a cylindrical surface of which the directrix is a straight line and the generatrix parallel to the X-axis; the locus of this equation is therefore a plane parallel to the X-axis.

If an equation contains only one Cartesian variable, its plane locus is a set of lines, real or complex, parallel to one of the coördi-

nate axes. Its space locus is therefore a set of planes, real or complex, parallel to one of the coördinate planes. For instance, the locus of the equation $x^2 - 7x + 12 = 0$ consists of two planes parallel to the YZ-plane, at distances of 3 and 4 units from the YZ-plane.

We recall that the geometric statement "a point P belongs to a certain locus" has as its algebraic equivalent "the coördinates of P satisfy the equation of a locus." With this fact in mind, we turn to a particularly interesting case of an equation in two variables, namely, the equation obtained by eliminating one of the variables from two equations in three variables. Suppose that we eliminate z from the equations $f(x, y, z) = 0$ and $g(x, y, z) = 0$, and that the result of the elimination is the equation $F(x, y) = 0$. What can be said about the loci of these three equations?

In the first place it follows from Theorem 1 that the locus of the equation $F(x, y) = 0$ is a cylindrical surface parallel to the Z-axis. On the other hand, since it is satisfied (provided the elimination has been carried out correctly) by all values of x, y, and z which satisfy the equations $f(x, y, z) = 0$ and $g(x, y, z) = 0$, it must pass through all the points common to the two surfaces which these two equations represent, that is, through their curve of intersection (see Definitions II and I). The locus of the equation $F(x, y) = 0$ is therefore the cylindrical surface which projects upon the XY-plane the curve represented by the equations $f(x, y, z) = 0$ and $g(x, y, z) = 0$, see Fig. 6. Moreover the plane locus of the equation $F(x, y) = 0$, consisting of the points in the XY-plane whose coördinates satisfy the equation, is clearly the projection of this curve on the XY-plane. A similar statement can be made concerning the equation obtained by eliminating x or y from the two given equations. We have obtained therefore the following theorem.

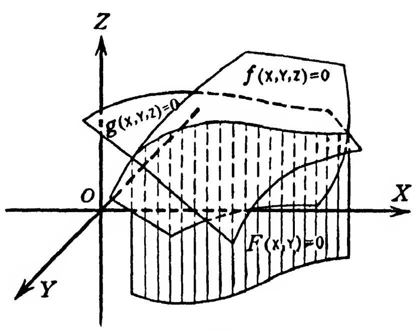

Fig. 6

THEOREM 2. **The equation in two variables, obtained by eliminating one variable from two equations in three variables, has as its space locus the cylindrical surface which projects the curve represented by the two equations upon the plane of the two remaining variables; and as its plane locus the projection of the curve on the same coördinate plane.**

41. The Linear Equation $ax + by + cz + d = 0$.

DEFINITION IV. **A *plane* is a set of points of such character that if any two points A and B belong to the set, then every point on the line joining AB also belongs to it.**

On the basis of this definition it is a simple matter to prove the following theorem.

THEOREM 3. **The locus of any equation of the first degree in x, y, and z is a plane.**

Proof. The most general equation of the first degree in x, y, and z is $ax + by + cz + d = 0$. If an arbitrary point P is taken on the line AB, its coördinates will be, according to the corollary to Theorem 9, Chapter II (see Section 34, page 58)

$$x_P = \frac{r_2x_A + r_1x_B}{r_1 + r_2}, \quad y_P = \frac{r_2y_A + r_1y_B}{r_1 + r_2}, \quad z_P = \frac{r_2z_A + r_1z_B}{r_1 + r_2},$$

when $r_1 : r_2 = AP : PB$. We have to show therefore that the identities $ax_A + by_A + cz_A + d \equiv 0$ and $ax_B + by_B + cz_B + d \equiv 0$ have as a consequence the identity

$$\frac{a(r_2x_A + r_1x_B)}{r_1 + r_2} + \frac{b(r_2y_A + r_1y_B)}{r_1 + r_2} + \frac{c(r_2z_A + r_1z_B)}{r_1 + r_2} + d \equiv 0.$$

If we write this equation in the equivalent form obtained by clearing it of fractions ($r_1 + r_2 \neq 0$) and collecting the terms in r_1 and those in r_2, namely,

$$r_2(ax_A + by_A + cz_A + d) + r_1(ax_B + by_B + cz_B + d) \equiv 0,$$

it should be indeed evident that it results from the two given identities. The theorem has therefore been proved.

Next we shall establish its converse.

THEOREM 4. **The equation of any plane is a linear equation in x, y, and z.**

Proof. We shall divide the proof into four parts.

(*a*) Suppose that the plane is parallel to two of the coördinate axes, that is, to one of the coördinate planes; let us say for the sake of definiteness that the plane is parallel to the ZY-plane at a distance k from it. Then for every point on the plane, and for no other points, the x-coördinate is equal to k. Therefore the equation of this plane is $x - k = 0$, which is obviously a linear equation.

(*b*) If the plane is parallel to one coördinate axis, for example, the Z-axis, let the line in which the plane meets the XY-plane, referred to the X- and Y-axes, have the equation $ax + by + c = 0$. It follows then that the equation of the plane is also $ax + by + c = 0$, see Theorem 1 and Remark 2 following it (Section 40, page 69); this is again a linear equation.

(*c*) Suppose that the plane cuts all three axes but does not pass through the origin. Let it cut the X-axis in the point $P(p, 0, 0)$, the Y-axis in the point $Q(0, q, 0)$, and the Z-axis in the point $R(0, 0, r)$; then p, q, and r are all three different from zero. And the given plane must be the locus of the equation $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1$. For this equation is linear; its locus is therefore a plane, by virtue of Theorem 3. Moreover, the points P, Q, and R clearly belong to this locus since their coördinates manifestly satisfy the equation. Hence the given plane and the plane which is the locus of this equation have three points in common; therefore they coincide.

(*d*) Suppose finally that the plane passes through the origin. We select in the plane two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ not collinear with O; and we seek to determine three numbers a, b, and c, not all zero, such that $ax_1 + by_1 + cz_1 = 0$ and $ax_2 + by_2 + cz_2 = 0$. Is this possible? Yes, for since the points O, P_1, and P_2 are not collinear, the coördinates x_1, y_1, z_1 are not proportional to the coördinates x_2, y_2, z_2 (see Corollary 3 to Theorem 6, Chapter III, Section 33, page 56 and Exercise 7, Section 31, page 53) and therefore not all the two-rowed minors of the matrix $\left\| \begin{matrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{matrix} \right\|$ can vanish, that is, the rank of this matrix is 2. But this fact enables us to conclude, by means of Theorem 4, Chapter II (Section 25, page 41) that we can indeed determine the numbers a, b, and c so as to satisfy the conditions mentioned above. The locus of the equation $ax + by + cz = 0$ has therefore the three

points O, P_1, and P_2 in common with the given plane; since this locus is moreover a plane (Theorem 3) it coincides with the given plane.

This completes the proof of the theorem.

Remark. 1 The directed distances p, q, and r from the origin to the points in which a plane cuts the coördinate axes are called the *intercepts* of the plane. The equation $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1$ which can be written down as soon as the intercepts of a plane are known, is usually referred to as the **intercept form of the equation of the plane.** It is a special case of the equation of the plane in terms of the coördinates of three of its points.

Corollary. **The intercepts of the plane which is the locus of the equation $ax + by + cz + d = 0$, in which neither a, nor b, nor c are equal to zero, are equal to $-\frac{d}{a}$, $-\frac{d}{b}$, and $-\frac{d}{c}$.**

Remark 2. It follows from this corollary that if d is a constant, different from zero and a, b, and c are variables which tend to zero, then the intercepts of the plane represented by the equation $ax + by + cz + d = 0$ increase beyond all bounds and hence the plane moves farther and farther from the origin — the usual phrase is "the plane moves to infinity." This is the sense in which we are to understand the statement that the equation $0 \cdot x + 0 \cdot y + 0 \cdot z + d = 0$ represents the "plane at infinity." A satisfactory treatment of the question here hinted at belongs in the field of Projective Geometry; we shall not undertake such treatment in this book. Whenever it becomes desirable to recognize explicitly that the plane under discussion is not the "plane at infinity" we shall speak of a **plane at finite distance.**

Theorem 5. **The plane which passes through the three non-collinear points P_1, P_2, and P_3 has the equation**

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

Proof. The required equation is linear in x, y, and z, by virtue of Theorem 4. If therefore $P(x, y, z)$ is any fourth point of the plane determined by the non-collinear points P_1, P_2, and P_3, there must exist four numbers a, b, c, and d which are not all zero, such

that

$$\begin{aligned} ax + by + cz + d &= 0, \\ ax_1 + by_1 + cz_1 + d &= 0, \\ ax_2 + by_2 + cz_2 + d &= 0, \\ \text{and} \qquad ax_3 + by_3 + cz_3 + d &= 0. \end{aligned}$$

This is a system of four linear homogeneous equations in the four variables a, b, c, and d. In order that this system of equations may possess a non-trivial solution, it is necessary, in view of Theorem 2, Chapter II and its corollary (Section 22, page 38), that the value of the coefficient determinant must be zero. Hence the coördinates of an arbitrary fourth point of the plane $P_1P_2P_3$ must satisfy the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

On the other hand this equation is linear in x, y, and z, in view of Theorem 12, Chapter I (Section 7, page 13). It is moreover satisfied by the coördinates of the points P_1, P_2, and P_3, as should be evident by use of Theorem 7, Chapter I (Section 5, page 9); therefore its locus is the plane determined by P_1, P_2, *and* P_3.

The equation established in this theorem is usually referred to as the **three-point form of the equation of the plane.**

42. Exercises.

1. How many equations are required to specify a curve in a space of four dimensions? To specify a surface in a four-space? To specify a three-space in a four-space?

2. Formulate a general statement of which the answers to the preceding exercise and the statements preceding the definitions in Section 39 are special cases.

3. Determine the loci in three-space of the following equations:

(*a*) $\frac{x^2}{4} - \frac{y^2}{9} = 1$ (*b*) $x^2 + y^2 + z^2 = 16$

(*c*) $4z = y^2$ (*d*) $x^2 - 5x + 6 = 0$

(*e*) $4x^2 + 6y^2 - 12 = 0$ (*f*) $z^3 - 6z^2 + 11z - 6 = 0$

4. Show that if P_1, P_2, and P_3 are collinear points, the equation in Theorem 5 is satisfied identically, that is, for all values of x, y, and z.

5. Show that, if three points lie on a plane which passes through the origin, the determinant whose rows are the coördinates of these points, all taken in the same order, has the value zero.

6. Determine the point on the line through the points $A(-4, 2, 5)$ and $B(1, -3, 2)$ which also lies on the plane determined by the points $P_1(0, 2, -1)$, $P_2(-6, -2, 0)$ and $P_3(-4, 0, 1)$.

7. Determine four points which lie on the plane $2x - 4y + z + 7 = 0$.

8. Determine four points which lie on the locus of the equation $3x^2 - 4y^2 + 5z^2 = 22$.

9. Determine three points which lie on the curve whose equations are

$$x + 2y - 3z = 5 \quad \text{and} \quad 2x - 3y + z = 3.$$

10. Determine three points on the curve which is the locus of the pair of equations

$$2x - y + 2z = 9 \quad \text{and} \quad x^2 + y^2 + z^2 = 26.$$

11. Determine the equations of the projections on each of the coördinate planes of the curve of the preceding exercise.

12. Determine the equations of the planes which pass through the following sets of three points each; find the intercepts of each of these planes:

(*a*) $P_1(1, 2, 3)$, $P_2(2, 3, 4)$, $P_3(3, 5, 7)$
(*b*) $P_1(-2, 3, 4)$, $P_2(-1, 2, 5)$, $P_3(7, 0, 2)$
(*c*) $P_1(3, -2, 5)$, $P_2(-2, 1, 3)$, $P_3(8, -3, 7)$
(*d*) $P_1(-4, 5, -2)$, $P_2(-4, 3, 1)$, $P_3(-4, -7, 3)$
(*e*) $P_1(2, 4, -5)$, $P_2(-3, 1, 2)$, $P_3(-5, 11, -4)$
(*f*) $P_1(3, -4, 2)$, $P_2(-2, -5, 1)$, $P_3(-1, -2, 4)$

13. Set up the condition which the coördinates of three points must sa in order that the plane determined by them shall be parallel to (*a*) the YZ-pl (*b*) the ZX-plane; (*c*) the XY-plane; the X-axis; the Y-axis; the Z-axi

14. Determine the conditions which the coördinates of three points m satisfy in order that they lie on a line.

43. The Distance from a Plane to a Point. To determine the distance from a plane to a point we make use of the projection method explained in Section 36; we divide the discussion into two parts.

(*a*) The plane does not pass through the origin.

Suppose that the direction cosines of the directed perpendicular *from the origin to the plane* are λ, μ, and ν and that the positive direction on this line is taken to be the direction from the origin to the plane.* Let the unsigned length of the distance from the origin

* This agreement as to the positive direction on the perpendicular from the origin to the plane is entirely arbitrary; if the opposite agreement were made, the interpretation of some of the results obtained in the following pages would be different, but equally useful. The convention adopted here is in accord with general practice. It would be a good exercise for the reader, after having thoroughly mastered the next few sections, to develop this part of the work on the basis of the opposite convention.

to the plane be designated by p and the foot of the perpendicular by H (see Fig. 7). Suppose furthermore that the given point is $P(\alpha, \beta, \gamma)$ and its projection on the given plane is Q.

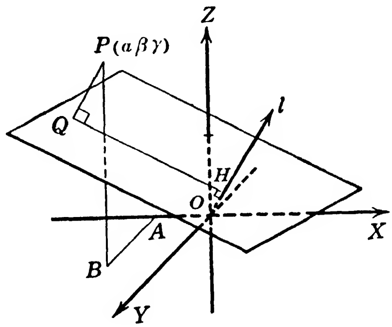

Fig. 7

We consider now the projection on the directed line l of the closed broken line which goes from O to P along the edges of the c.p. of and from P back to O by way of Q and H. By virtue of Theorem Chapter III (Section 36, page 62), we find that

$$\text{Proj}_l OA + \text{Proj}_l AB + \text{Proj}_l BP + \text{Proj}_l PQ + \text{Proj}_l QH + \text{Proj}_l HO = 0.$$

If we evaluate these projections by means of Theorem 11, Chapter III (Section 36, page 62), noticing that $QP \parallel l$ and $QH \perp l$, we conclude that

$$\alpha\lambda + \beta\mu + \gamma\nu + PQ + 0 + HO = 0.$$

Here we have to bear in mind that the sign of PQ is to be taken in accordance with the direction specified on l, also that $HO = -p$. Accordingly we obtain the result given in the following theorem.

Theorem 6. **The distance from a plane which does not pass through the origin and for which the perpendicular directed from the origin to the plane has direction cosines λ, μ, and ν, and unsigned length p, to the point $P(\alpha, \beta, \gamma)$ is equal to**

$$QP = \alpha\lambda + \beta\mu + \gamma\nu - p.$$

(*b*) In case the plane passes through the origin, the specification of the positive direction on the line l becomes meaningless; but, if we designate by λ, μ, ν the direction cosines of either direc-

tion on a perpendicular to the plane, the proof goes through as in part (*a*). We conclude therefore, since now $p = 0$, that the distance QP from a plane through the origin to the point $P(\alpha, \beta, \gamma)$ is equal to $\alpha\lambda + \beta\mu + \gamma\nu$, where λ, μ and ν are the direction cosines of either direction on a perpendicular to the plane.

COROLLARY 1. **The unsigned distance from a plane through the origin to the point $P(\alpha, \beta, \gamma)$ is equal to the numerical value of $\alpha\lambda + \beta\mu + \gamma\nu$, where λ, μ, and ν are the direction cosines of either direction on a perpendicular to the plane.**

Remark. It follows from the above discussion that if the distance QP from a plane not through the origin to P turns out to be positive, then the direction of QP agrees with the positive direction along l, that is, P lies on the side of the plane opposite to that on which the origin lies; whereas, if the distance QP turns out to be negative, P lies on the same side of the plane as the origin. Furthermore, if for a plane through the origin, the expression $\alpha\lambda + \beta\mu + \gamma\nu$ turns out to be positive (negative), the point P lies on the side of the plane (on the side opposite to that) indicated by the direction which the direction cosines λ, μ, ν specify.

If, whether the plane passes through the origin or not, the distance from the plane, calculated by means of Theorem 6 or Corollary 1, turns out to be zero, the point P lies on the plane. And conversely, it should be clear that if P lies on the plane, its distance from the plane is zero. This simple fact enables us to state an important further corollary of the theorem:

COROLLARY 2. **The coördinates x, y, z of a point P on a plane satisfy the equation**

$$\lambda x + \mu y + \nu z - p = 0.$$

If the plane does not pass through the origin, then λ, μ, and ν are the direction cosines of the perpendicular directed from the origin to the plane and p is the unsigned distance from the origin to the plane; if the plane passes through the origin, then $p = 0$ and λ, μ, ν are the direction cosines of either direction on a perpendicular to the plane.

44. The Normal Form of the Equation of a Plane. A comparison of the equation established in Corollary 2 of the preceding section with the general linear equation in x, y, and z yields a

valuable result. For we have seen in Theorem 4 (Section 41, page 71) that every plane can be represented by an equation of the form $ax + by + cz + d = 0$, in which a, b, c, and d had no particular significance; and in Corollary 2 of Section 43 we established the fact that every plane can be represented by an equation of the form $\lambda x + \mu y + \nu z - p = 0$, in which λ, μ, ν, and p have the geometrical meanings stated in this corollary. But if these two equations represent the same plane, they must be equivalent; hence their coefficients must be proportional; thus there exists a non-vanishing number k such that

$$a = k\lambda, b = k\mu, c = k\nu \quad \text{and} \quad d = -kp.^*$$

From the first three of these equations we conclude (see Theorem 8, Chapter III, Section 33, page 56) that $k = \pm\sqrt{a^2 + b^2 + c^2}$; from the last it follows, since p is an unsigned number, that the sign of k must be opposite to that of d. Thus k is completely determined if the plane does not pass through the origin, whereas its sign is left ambiguous if the plane passes through the origin. We have therefore obtained the following geometrical interpretation of the coefficients in the general linear equation in x, y, and z.

THEOREM 7. **The coefficients a, b, and c of the variables x, y, z in the equation of a plane, $ax + by + cz + d = 0$, are proportional to the direction cosines of a line perpendicular to the plane; if $d \neq 0$, the quotients of a, b, and c by that square root of the sum of their squares which is opposite in sign to d, are equal to the direction cosines of the perpendicular directed from the origin to the plane, and the quotient of $-d$ by the same square root gives the unsigned distance from the origin to the plane.**

Remark 1. Corollary 2 of Section 43 gives us another form in which the equation of a plane may be written. It is called the **normal form of the equation of a plane.** This form of the equation of the plane is characterized by the two facts that the sum of the squares of the coefficients of the variables is equal to 1 and that the constant term is negative or zero.

Remark 2. Division of the form $ax + by + cz + d = 0$ of the equation of a plane by $+\sqrt{a^2 + b^2 + c^2}$ or by $-\sqrt{a^2 + b^2 + c^2}$

* Two equations are equivalent if any values of the variables which occur in it that satisfy either one of them also satisfy the other. It is a nice exercise in algebra to show that, if two linear equations in x, y, and z are equivalent, their coefficients are proportional.

according as d is negative or positive is called "reduction of the equation of the plane to the normal form."

If we combine Theorems 7 and 6, we obtain the following corollaries.

COROLLARY 1. **The distance *from* the plane $ax + by + cz + d = 0$, $d \neq 0$, *to* the point $P(\alpha, \beta, \gamma)$ is equal to $\dfrac{a\alpha + b\beta + c\gamma + d}{\pm\sqrt{a^2 + b^2 + c^2}}$, the + or − sign being used according as d is negative or positive; this distance will be positive or negative according as P and the origin lie on opposite sides or on the same side of the plane.**

COROLLARY 2. **The unsigned distance from a plane through the origin $ax + by + cz = 0$ to the point $P(\alpha, \beta, \gamma)$ is given by the numerical value of $\dfrac{a\alpha + b\beta + c\gamma}{\sqrt{a^2 + b^2 + c^2}}$.**

If a definite choice of the sign of the square root in this last formula is determined upon, then those points P for which the distance turns out to be positive (negative) lie on the side of the plane (on the side opposite to that) indicated by the direction whose direction cosines are equal to the quotients of a, b, and c divided by that square root. Although in this case the parts of space on opposite sides of the plane are not so readily characterized as when the plane does not pass through the origin, we still have this essential fact that, once the sign of the square root has been fixed in either way, two points $P(\alpha, \beta, \gamma)$ and $P'(\alpha', \beta', \gamma')$ will lie on the same or on opposite sides of the plane according as $\dfrac{a\alpha + b\beta + c\gamma}{\sqrt{a^2 + b^2 + c^2}}$ and $\dfrac{a\alpha' + b\beta' + c\gamma'}{\sqrt{a^2 + b^2 + c^2}}$ are equal or opposite in sign, that is, according as $a\alpha + b\beta + c\gamma$ and $a\alpha' + b\beta' + c\gamma'$ have the same or opposite signs.

From Theorem 7, in combination with Theorem 13, Chapter III (Section 36, page 63) we obtain moreover the following result.

COROLLARY 3. **The angles θ between a line whose direction cosines are λ, μ, ν and the plane $ax + by + cz + d = 0$ are determined by the equation**

$$\sin\theta = \frac{\lambda a + \mu b + \nu c}{\pm\sqrt{a^2 + b^2 + c^2}}.$$

The angle between a line and a plane is the angle between that line and its projection on the plane; the sine of this angle is therefore equal to the cosine of the angle made by the given line and a

line perpendicular to the plane. Corollary 3 follows from these observations. The ambiguity of sign corresponds to the fact that neither the direction on the given line nor that on the projection have been specified.

Examples.

1. To find the distance from the plane Π: $2x + 3y - 4z + 5 = 0$ to the points $A(-1, 2, 4)$, $B(3, -2, 0)$, $O(0, 0, 0)$ and $C(3, 3, 5)$, we determine the direction cosines of the perpendicular directed from the origin to the plane; it is found that $\lambda = \dfrac{2}{-\sqrt{29}}$, $\mu = \dfrac{3}{-\sqrt{29}}$, $\nu = \dfrac{-4}{-\sqrt{29}}$. The unsigned length of the perpendicular from the origin to the plane is $\dfrac{5}{\sqrt{29}}$. It follows that the distance $\Pi A = \dfrac{-2 + 6 - 16 + 5}{-\sqrt{29}} = \dfrac{7\sqrt{29}}{29}$, that $\Pi B = \dfrac{6 - 6 - 0 + 5}{-\sqrt{29}} = \dfrac{-5\sqrt{29}}{29}$, that $\Pi O = \dfrac{-5\sqrt{29}}{29}$ and that $\Pi C = \dfrac{6 + 9 - 20 + 5}{-\sqrt{29}} = 0$. We conclude that A and O lie on opposite sides, B and O on the same side of the plane, while C is on the plane. These are the geometrically essential facts concerning the positions of these points and the plane; that the side of the plane on which the origin lies happens to be the negative side is not of geometric importance, but is a result of the convention made in Section 43 (see the footnote on page 75).

2. To find the distances from the plane $\Pi : 3x - 12y + 4z = 0$ to the points $A(-3, 1, 4)$, $B(3, -12, 4)$ $C(-3, 12, -4)$ and $D(5, 1, 1)$, we determine the direction cosines of the line perpendicular to the plane. We find that $\lambda : \mu : \nu = 3 : -12 : 4$, so that $\lambda = \pm\frac{3}{13}$, $\mu = \mp\frac{12}{13}$, $\nu = \pm\frac{4}{13}$, in which either all the upper signs or all the lower signs are to be used. If we choose the upper signs, we find that $\Pi A = \dfrac{-9 - 12 + 16}{13} = -\frac{5}{13}$, $\Pi B = +13$, $\Pi C = -13$ and $\Pi D = \frac{7}{13}$, from which we conclude that A and C lie on the side of the plane opposite to that indicated by the direction whose direction cosines are $\frac{3}{13}$, $-\frac{12}{13}$, $\frac{4}{13}$, but B and D lie on the side indicated by that direction. If we choose the lower signs, the signs of the four distances are reversed; this means that A and C lie on the side of the plane indicated by the direction whose direction cosines are $-\frac{3}{13}$, $\frac{12}{13}$, $-\frac{4}{13}$, and B and D lie on the opposite side. These conclusions are obviously identical in geometrical content with those stated in the preceding sentence.

45. Exercises.

1. Determine the distances of the points $A(-3, 2, 1)$, $B(5, -3, -1)$, $C(2, 4, 2)$ and $D(-1, 2, -4)$ from the plane $3x + 2y - 6z - 2 = 0$, and determine their positions relative to the plane.

2. Also the distances of the points $A(1, 4, -3)$, $B(3, -2, 2)$, $C(-5, 1, 3)$ and $D(1, 0, 2)$ from the plane $2x - 3y + z = 0$.

3. Find the direction cosines of the lines which are perpendicular to the following planes:

(a) $14x - 3y + 18z + 1 = 0$ (c) $6x - 2y - 3z + 2 = 0$
(b) $2x + 3y - 2z + 4 = 0$ (d) $x + 4y - 8z - 3 = 0$.

4. Determine the distances:

(a) from the X-axis to the plane $3y - 4z + 7 = 0$
(b) from the Y-axis to the plane $5x - 2z - 3 = 0$
(c) from the Z-axis to the plane $5x - 12y - 8 = 0$.

5. Determine the coördinates of the point in which the plane $2x - y - 2z + 4 = 0$ is met by a line through $A(-3, 1, 2)$ perpendicular to the plane. Find the distance from the plane to A in two ways.

6. Find the coördinates of the point in which the plane $ax + by + cz + d = 0$ is met by the perpendicular from $P(\alpha, \beta, \gamma)$ to the plane.

7. Through the point $A(2, -2, 6)$ in the plane $3x + 2y - z + 4 = 0$, a line is drawn whose direction cosines are proportional to 3, -6 and 2. Find the angles which this line makes with the plane.

8. Set up the equation of a plane through the point $A(-2, 3, 1)$ and perpendicular to a line whose direction cosines are proportional to -3, 4, 2.

46. Two Planes. Two distinct planes either intersect in a line or else they are parallel. Since two planes are perpendicular to the same line if and only if they are parallel, it follows immediately from Theorem 7 that two planes

(1) $a_1x + b_1y + c_1z + d_1 = 0$ and (2) $a_2x + b_2y + c_2z + d_2 = 0$

are distinct and parallel if and only if $a_1 : a_2 = b_1 : b_2 = c_1 : c_2 \neq d_1 : d_2$, that is, if the rank of the coefficient matrix of the two equations is 1, and the rank of the augmented matrix is 2 (see Definition IX, Chapter I, Section 9, page 16 and Section 20, last paragraph). If two equations represent the same plane, their coefficients are proportional (see footnote on page 78), the two-rowed minors of the augmented matrix all vanish and the rank of the a.m. is 1. We can therefore state the following conclusion.

THEOREM 8. **The planes represented by two linear equations are (1) coincident if and only if the rank of the augmented matrix is 1; (2) parallel if and only if the rank of the augmented matrix is 2 and the rank of the coefficient matrix is 1; (3) intersecting if and only if the rank of the coefficient matrix is 2.**

To determine the angles between two intersecting planes, we make use once more of Theorem 7. These angles are the same as the angles between two lines perpendicular to these planes

(see Fig. 8). Therefore the angles between the planes (1) and (2) are equal to the angles between the lines whose direction cosines are proportional to a_1, b_1, c_1 and to a_2, b_2, c_2. Consequently, if θ is

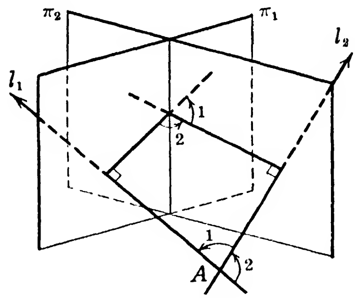

FIG. 8

used to designate any one of these angles, we conclude, using also Theorem 13, Chapter III (Section 36, page 63), that

$$(3) \quad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{(\pm\sqrt{a_1^2 + b_1^2 + c_1^2})\,(\pm\sqrt{a_2^2 + b_2^2 + c_2^2})}.$$

THEOREM 9. **The cosine of any of the angles between the planes represented by two linear equations is equal to the sum of the products of the coefficients of the like variables in the two equations, divided by the product of the square roots of the sums of the squares of these coefficients.**

Remark 1. The ambiguity of sign in the formula corresponds to the fact that the different angles formed by two planes are related in such a way that their cosines differ at most in sign.

Remark 2. If the square roots in the denominator of formula (3) are given the signs opposite to those of d_1 and d_2 respectively, we obtain the cosine of the angle between the perpendiculars to the plane, directed in each case from the origin to the plane (see Theorem 7, page 78), that is, the cosine of the supplement of that angle between the planes in which the origin lies. We are supposing in this statement that neither plane passes through the origin.

COROLLARY. **Two planes, $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, are perpendicular if and only if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.**

An equation of the form $(a_1x + b_1y + c_1z + d_1)(a_2x + b_2y + c_2z + d_2) = 0$ is satisfied by values of the variables which cause

at least one of the factors of its left-hand side to vanish, and by such values only. The locus of this equation consists therefore of the two planes represented by the equations $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$. It should be clear that this observation may be generalized as in the following theorem.

THEOREM 10. **If $F(x, y, z) \equiv f_1(x, y, z) \cdot f_2(x, y, z) \cdot \ldots \cdot f_k(x, y, z)$, the locus of the equation $F(x, y, z) = 0$ consists of the loci of the equations $f_1(x, y, z) = 0, f_2(x, y, z) = 0, \ldots, f_k(x, y, z) = 0$.**

DEFINITION V. **If the function $F(x, y, z)$ is factorable into real factors (that is factors which involve only real operations on the variables), the locus of the equation $F(x, y, z) = 0$ is called a *degenerate locus*.**

47. The Line. The coördinates of every point on the line of intersection of the two planes represented by equations (1) and (2) of the preceding section satisfy, in case (3) of Theorem 8, these two equations. Conversely, every point whose coördinates satisfy these equations lies on the line of intersection of the planes. We say therefore, in accordance with Definition II (Section 39, page 68), that "two linear equations $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, whose coefficient matrix has rank 2, are **the equations of a line.**"

Remark. A line, thus defined as the intersection of two planes, has as its equations those of two planes passing through it. But there is a single infinitude of planes which pass through a given straight line (see Section 49) and the equations of any two of these planes can be taken as the equations of the line. Thus it is seen that one and the same straight line can be represented by any one of an infinite number of pairs of linear equations. The reader may at first be troubled by this lack of definiteness; he will do well to think this question through until it has become clear to him.

The results obtained in Chapter III, where the line was discussed as a locus of points, can now be interpreted in the light of the point of view presented in the first paragraph of this section. The equations found in Theorems 9 and 10, and in Corollary 1 of Theorem 10, Chapter III (see Section 34, pages 57 and 59) are all linear equations; and it is readily seen that in each case a pair of equations can be selected whose coefficient matrix has rank 2. For example, the equations of Theorem 9 may be written in the

following form:

$$\begin{aligned}
(y_B - y_A)x - (x_B - x_A)y \qquad\qquad\qquad\qquad &+ y_Ax_B - y_Bx_A = 0,\\
(z_B - z_A)y - (y_B - y_A)z &+ z_Ay_B - z_By_A = 0, \text{ and}\\
(z_B - z_A)x \qquad\qquad\qquad - (x_B - x_A)z &- x_Az_B + x_Bz_A = 0.
\end{aligned}$$

The second order determinants of the coefficient matrix of the first two of these equations have the values

$$(x_B - x_A)(y_B - y_A),\ -(y_B - y_A)^2,\ (y_B - y_A)\ (z_B - z_A);$$

those formed from the coefficient matrix of the second and third equations have the values

$$-(x_B - x_A)\ (z_B - z_A),\ (y_B - y_A)\ (z_B - z_A),\ -(z_B - z_A)^2;$$

and those obtained from the third and first equations have the values

$$-(x_B - x_A)^2,\ (y_B - y_A)\ (x_B - x_A),\ -(x_B - x_A)\ (z_B - z_A)\,.$$

Now it should be clear that if A and B are distinct points at least one of these second order determinants must have a value which is different from zero; therefore a pair of equations can be selected whose coefficient matrix has rank 2.

Similar arguments can be made for the equations of Theorem 10 and Corollary 1 of Theorem 10. A mere restatement of the earlier results in terms of the terminology which was introduced and justified at the beginning of the present section, leads to the following theorems.

THEOREM 11. **The equations of the straight line which passes through the points A and B may be written in the form:**

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A};$$

also in the forms:

$$\frac{x - x_B}{x_B - x_A} = \frac{y - y_B}{y_B - y_A} = \frac{z - z_B}{z_B - z_A},$$

or

$$\frac{x - x_A}{x_B - x} = \frac{y - y_A}{y_B - y} = \frac{z - z_A}{z_B - z}.$$

THEOREM 12. **The equations of the directed line which passes through A and whose direction cosines are λ, μ and ν may be written in the form:**

$$\frac{x - x_A}{\lambda} = \frac{y - y_A}{\mu} = \frac{z - z_A}{\nu}.$$

THEOREM 13. **The equations of the undirected line which passes through the point A and whose direction cosines are proportional to l, m and n may be written in the form:**

$$\frac{x - x_A}{l} = \frac{y - y_A}{m} = \frac{z - z_A}{n}.$$

Remark 1. It should be clear that in each of these theorems the line is the intersection of three planes, any two of which suffice to determine it. The three planes are, in each case, parallel to the X-, Y-, and Z-axes; they are indeed the planes which project the line on the three coördinate planes (see Fig. 9), the diagonal planes of the c.p. of any two points on the line.

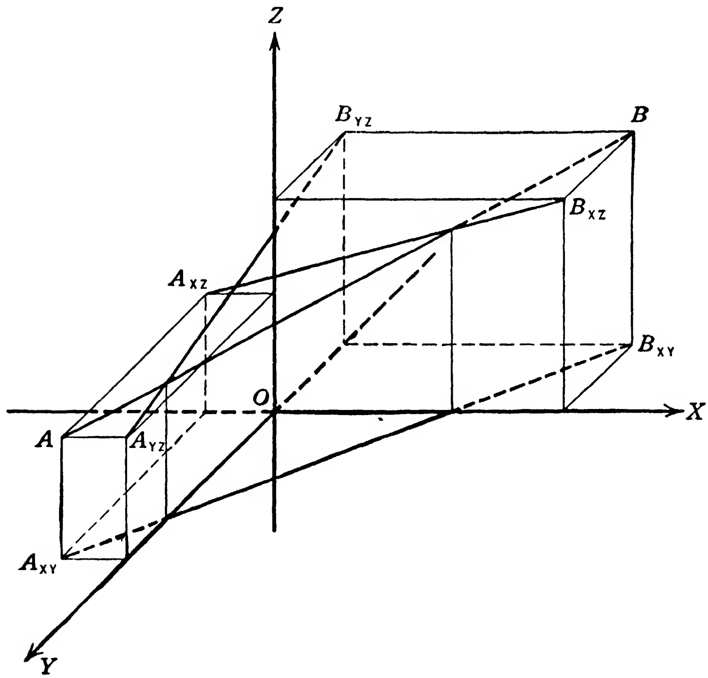

FIG. 9

Remark 2. The equations of the line, established in ~~Theorems~~ 11, 12, and 13, lose meaning whenever one of the denominators vanishes. In spite of this disadvantage these forms for the equations of the line are, in general, more convenient than the extended form obtained by equating two of the fractions at a time and then

clearing of fractions. This extended form becomes imperative if one of the denominators vanishes. Frequently one finds the condensed form used even in such cases; this is however not to be recommended even though this apparently meaningless form is intended to be symbolic for the extended equations. The difficulty referred to here can be obviated by use of the parametric equations, already obtained in Chapter III (see the Corollary of Theorem 9, and Corollaries 2 and 3 of Theorem 10, Section 34, pages 58, and 60), whose existence is formulated again as follows.

THEOREM 14. **The equations of the line through the points A and B may be written in the following form:**

$$x = \frac{x_A + rx_B}{1 + r}, \quad y = \frac{y_A + ry_B}{1 + r}, \quad z = \frac{z_A + rz_B}{1 + r}.$$

THEOREM 15. **The equations of the directed line through the point A whose direction cosines are equal to λ, μ, and ν may be written in the form:**

$$x = x_A + \lambda s, \quad y = y_A + \mu s, \quad z = z_A + \nu s.$$

THEOREM 16. **The equations of the undirected line through the point A whose direction cosines are proportional to l, m and n may be put in the form:**

$$x = x_A + lt, \quad y = y_A + mt, \quad z = z_A + nt.$$

Remark. In each of the last three theorems the line is given by means of three equations; but these equations involve four variables, namely the coördinates x, y, and z of the variable point along the line, *and* the parameter r, s, or t. The locus of these sets of equations has therefore one degree of freedom (see Section 39, page 67). The geometric significance of the parameters r, s, and t was discussed in Corollaries 2 and 3 of Theorem 10, Chapter III, and in Remark 4 following the Corollary of Theorem 9 (see Section 34, page 58); it is desirable that the reader recall this interpretation of the parameters at this point.

If an undirected line is given by means of two linear equations in the general form, like equations (1) and (2) of Section 46, whose coefficient matrix has rank 2, these equations can be reduced to any one of the forms given in Theorems 11 to 16, as soon as the coördinates of two points on the line and the ratios of its direction cosines have been determined.

Since the coefficient matrix is of rank 2, the equations can be solved for two of the variables in terms of the third. By assigning values to this third variable arbitrarily, an infinite number of solutions of the equations can be obtained; but each of these solutions furnishes the coördinates of a point on the given line.

The direction cosines of the line are found by means of the following theorem:

THEOREM 17. **The direction cosines of the line of intersection of two intersecting planes are proportional to the two-rowed minors of the coefficient matrix of their equations, taken alternately with the plus and the minus signs.**

Proof. The proof of this very useful theorem can be made in various ways. We shall make use here of Corollary 1 of Theorem 6, Chapter III (Section 33, page 56). Suppose that P_1 and P_2 are two arbitrary points on the line of intersection of the planes. Then

$$a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0, \qquad a_2x_1 + b_2y_1 + c_2z_1 + d_2 = 0,$$
$$a_1x_2 + b_1y_2 + c_1z_2 + d_1 = 0 \quad \text{and} \quad a_2x_2 + b_2y_2 + c_2z_2 + d_2 = 0.$$

If we subtract these equations in pairs, we find that the differences of the coördinates of P_1 and P_2 satisfy the following two linear homogeneous equations:

$$a_1(x_1 - x_2) + b_1(y_1 - y_2) + c_1(z_1 - z_2) = 0 \quad \text{and} \quad a_2(x_1 - x_2) + b_2(y_1 - y_2) + c_2(z_1 - z_2) = 0.$$

Since the two given planes intersect, it follows from (3) in Theorem 8 (Section 46, page 81) that the rank of the coefficient matrix of these equations is 2. Theorem 4, Chapter II (Section 25, page 41) gives us the means therefore to determine from these equations the ratios of the coördinate differences of P_1 and P_2. But we know from Corollary 1 of Theorem 6, Chapter III (Section 33, page 56) that these coördinate differences are proportional to the direction cosines λ, μ, ν of the line. We find therefore that

$$\lambda : \mu : \nu = x_1 - x_2 : y_1 - y_2 : z_1 - z_2 =$$
$$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

This completes the proof of the theorem.

Examples.

1. The planes represented by the equations $2x - y + 3z - 4 = 0$ and $2x - y + 5z + 3 = 0$ intersect in a line; for the rank of the matrix

$$\begin{Vmatrix} 2 & -1 & 3 \\ 2 & -1 & 5 \end{Vmatrix}$$

is clearly 2. To determine points on the line of intersection, we solve the equations for x and z in terms of y (we could equally well solve them for y and z in terms of x, but not for x and y in terms of z; why not?) by Cramer's rule. We find

$$x = \frac{2y + 29}{4}, \quad z = -\frac{7}{2}.$$

By selecting values for y and calculating the corresponding values of x and z from these equations, we can find as many points on the line as we wish; thus we locate the points $A(8, \frac{3}{2}, -\frac{7}{2})$, $B(\frac{25}{4}, -2, -\frac{7}{2})$ and $C(\frac{29}{4}, 0, -\frac{7}{2})$ on the line of intersection of the given planes. Having determined these points, we can find the direction cosines of the line most simply by use of Corollary 1 of Theorem 6, Chapter III directly; it is found that

$$\lambda : \mu : \nu = 8 - \tfrac{25}{4} : \tfrac{3}{2} + 2 : -\tfrac{7}{2} + \tfrac{7}{2} = \tfrac{7}{4} : \tfrac{7}{2} : 0 = 1 : 2 : 0.$$

Consequently $\lambda = \dfrac{1}{\pm\sqrt{5}}$, $\mu = \dfrac{2}{\pm\sqrt{5}}$, $\nu = 0$, so that the line makes an angle of $90°$ with the Z-axis and is therefore parallel to the XY-plane; this could have been foretold from the fact that all of its points have the same z-coördinate, $-\frac{7}{2}$.

The ratios of the direction cosines can also be found by applying the formula proved in Theorem 17; this gives us

$$\lambda : \mu : \nu = \begin{vmatrix} -1 & 3 \\ -1 & 5 \end{vmatrix} : -\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} : \begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix} = -2 : -4 : 0 = 1 : 2 : 0.$$

In accordance with Remark 2, following Theorem 13 (page 85), the non-parametric forms of the equations of the line as given in Theorems 11, 12, and 13 are not desirable in this case. The parametric forms of the equations are:

$$x = 8 + t,\ y = \tfrac{3}{2} + 2t,\ z = -\tfrac{7}{2}, \quad \text{where} \quad t = \frac{AP}{\pm\sqrt{5}};$$

$$x = \frac{\frac{25}{4} + 8r}{1 + r}, \quad y = \frac{-2 + \frac{3}{2}r}{1 + r}, \quad z = -\tfrac{7}{2}, \quad \text{where} \quad r = \frac{PB}{AP};$$

$$x = \frac{29}{4} \pm \frac{s}{\sqrt{5}}, \quad y = \frac{\pm 2s}{\sqrt{5}}, \quad z = -\tfrac{7}{2}, \quad \text{where} \quad s = AP.$$

We observe that the three sets of values of x, y, and z given by the different parametric equations satisfy the equations of the two given planes identically in t, r, or s respectively.

2. The planes represented by the equations $2x - y + 3z - 4 = 0$ and $4x - 2y + 6z + 3 = 0$ are parallel; for the rank of the c.m. is 1 and the rank of the a.m. is 2. The parallelism or coincidence of two planes can readily be recognized upon inspection of the equations.

3. The direction cosines of the line of intersection of the planes represented by the equations $10x + 3y - 4z + 8 = 0$ and $4x + 3y - 3z - 4 = 0$ are proportional to $bc_1 : a_1c_1 : a_1b$

$$\begin{vmatrix} 3 & -4 \\ 3 & -3 \end{vmatrix} : -\begin{vmatrix} 10 & -4 \\ 4 & -3 \end{vmatrix} : \begin{vmatrix} 10 & 3 \\ 4 & 3 \end{vmatrix}; \quad \text{that is,} \quad \lambda : \mu : \nu = 3 : 14 : 18.$$

Since $3^2 + 14^2 + 18^2 = 529 = 23^2$, it follows that $\lambda = \pm\frac{3}{23}$, $\mu = \pm\frac{14}{23}$, $\nu = \pm\frac{18}{23}$.

Solution of the equations for x and y in terms of z leads to $x = \frac{z}{6} - 2$, $y = \frac{7z}{9} + 4$; we are now able to determine readily as many points on the line as we wish.

The parametric equations of the line may be written in the following forms:

$$x = -2 + \frac{3s}{23}, \quad y = 4 + \frac{14s}{23}, \quad z = \frac{18s}{23};$$

$$x = -2 + 3t, \quad y = 4 + 14t, \quad z = 18t;$$

$$x = \frac{-2 + 7r}{1 + r}, \quad y = \frac{4 + 46r}{1 + r}, \quad z = \frac{54r}{1 + r}.$$

We can verify that these values of x, y, and z satisfy the given equations of the planes identically.

The angles between the two planes are given by the equation:

$$\cos\theta = \frac{10 \cdot 4 + 3 \cdot 3 + (-4)(-3)}{\pm\sqrt{10^2 + 3^2 + (-4)^2} \times \pm\sqrt{4^2 + 3^2 + (-3)^2}} = \frac{61}{\pm 5\sqrt{170}}.$$

The direction cosines of the directed lines from the origin perpendicular to the plane are equal to $-\frac{2}{\sqrt{5}}, -\frac{3}{5\sqrt{5}}, \frac{4}{5\sqrt{5}}$ and $\frac{4}{\sqrt{34}}, \frac{3}{\sqrt{34}}, -\frac{3}{\sqrt{34}}$. Therefore the cosine of that angle between the planes in which the origin lies is equal to $-\frac{61}{5\sqrt{170}}$.

48. Exercises.

1. Determine by inspection which of the following pairs of equations represent intersecting, which parallel planes, and which coincident planes:

(*a*) $3x - y + 4z + 1 = 0$ and $2x + y - 2z + 3 = 0$
(*b*) $2x + y - 3z + 4 = 0$ and $2x + y - 3z - 4 = 0$
(*c*) $x + 2y + 4z - 3 = 0$ and $x - 2y + 4z + 1 = 0$
(*d*) $x - y + z = 0$ and $2x - 2y + 2z + 7 = 0$

2. Determine the angles between the planes of the intersecting pairs of planes in the preceding exercise.

3. Determine the distances between the following pairs of parallel planes:

(*a*) $x - 8y + 4z - 3 = 0$ and $x - 8y + 4z + 15 = 0$;
(*b*) $2x - 3y - 6z + 5 = 0$ and $2x - 3y - 6z + 19 = 0$;
(*c*) $x + y + z + 6 = 0$ and $x + y + z - 8 = 0$.

4. Write the equations of the lines which pass through the following pairs of points:

(*a*) $A(-3, 5, 2)$ and $B(5, 4, -2)$, (*c*) $A(5, 2, -3)$ and $B(-1, -1, -1)$,
(*b*) $A(4, -3, 1)$ and $B(-8, 3, 5)$, (*d*) $A(-2, 4, 1)$ and $B(3, -5, 2)$.

5. Write the equations of the line through $A(3, 4, -1)$ and perpendicular to the plane $2x - y + 2z - 5 = 0$. Determine the coördinates of the point in which the plane is met by this line.

6. Set up the equation of the plane through the point $A(-2, -3, 4)$ and
(*a*) parallel to the plane $3x + y - 5z + 7 = 0$;
(*b*) perpendicular to the line $\dfrac{x}{4} = \dfrac{y-3}{6} = \dfrac{z+2}{-12}$.

7. Determine the parametric equations of

(*a*) the line of intersection of the planes $3x - y + 3z - 2 = 0$ and $x + 2y - 3z + 4 = 0$;
(*b*) the line through the point $A(1, -3, 5)$ and parallel to the line of intersection of the planes $3x + y + 2z - 3 = 0$ and $6x + 3y + 2z + 5 = 0$.

8. Determine a plane through the points $A(2, 1, -4)$ and $B(-1, 3, 2)$, which intersects the plane $3x - y - 2z = 4$ in a line that makes equal angles with the coördinate axes.

9. Find the equation of a plane through the points $A(1, 3, -2)$ and $B(2, -4, 5)$ which is perpendicular to the plane $3x + 6y - 4z - 5 = 0$.

10. Find the parametric equations of a line through the point $A(2, -5, 3)$ and parallel to the line of intersection of the two planes represented by the equation $x^2 - 6xy + 9y^2 - 4z^2 + 12z - 9 = 0$.

11. Set up the equation of a plane through the point $A(-1, -4, 3)$ and perpendicular to the line of intersection of the two planes represented by the equation $9x^2 - 4y^2 + z^2 - 6xz - 4y - 1 = 0$.

12. Find the equation of a plane through the point $A(3, 2, -1)$ and perpendicular to the two planes $x - y + z + 4 = 0$ and $2x - y - 2z + 3 = 0$.

13. Determine the coördinates of the point in which the plane $3x - 4y + z - 3 = 0$ is met by the line $x = 2 - 2t$, $y = -1 - 3t$, $z = 5 + t$.

14. Determine the coördinates of the point in which the plane $4x + y - 3z + 5 = 0$ is met by the line which joins the points $A(-1, 3, -2)$ and $B(4, -3, 1)$.

49. The Pencil of Planes. The Bundle of Planes. If the left-hand sides of the equations

$$(1) \quad a_1x + b_1y + c_1z + d_1 = 0$$

and

$$(2) \quad a_2x + b_2y + c_2z + d_2 = 0$$

are denoted by E_1 and E_2 respectively* and if k_1 and k_2 are arbitrary constants, then the equation

$$(3) \quad k_1E_1 + k_2E_2 = 0$$

is also a linear equation. Its locus is therefore a plane. Moreover equation (3) will be satisfied by the coördinates of those points which lie on *both* planes E_1 and E_2, that is, by the points on the line of intersection of these planes; and this last statement holds true whether k_1 and k_2 are constants or not, because the coördinates of the points on the line of intersection of the planes cause both E_1 and E_2 to vanish. Consequently, the equation represents a plane through the line of intersection of the planes E_1 and E_2 for any constant values assigned to k_1 and k_2.

On the other hand the equation of every plane through this line can, by suitable choice of the values to be given to the constants k_1 and k_2, be put in the form (3). We see, in particular, that for $k_1 = 1$ and $k_2 = 0$ we obtain the plane E_1; and for $k_1 = 0$ and $k_2 = 1$, we obtain the plane E_2. And if any other plane through the line of intersection, l, of the two planes is given, and if $P(\alpha, \beta, \gamma)$ is an arbitrary point in such a plane but not on l, then equation (3) will represent the given plane, provided k_1 and k_2 are so chosen that

$$k_1(a_1\alpha + b_1\beta + c_1\gamma + d_1) + k_2(a_2\alpha + b_2\beta + c_2\gamma + d_2) = 0.$$

Since P does not lie on l and hence not on both planes, the two expressions in the parentheses do not both vanish; consequently the ratio $k_1 : k_2$ can always be determined in such a manner that the last written equation is satisfied. If numbers which have this ratio are substituted for k_1 and k_2 in equation (3), this equation will indeed have the given plane as its locus. If we introduce now

* When this abbreviated notation is employed for the left-hand side of the linear equation in x, y, and z, it is usually convenient to use the same letter to designate the plane which is the locus of the equation. Thus we shall speak of "the plane E_1" instead of using the longer and more explicit phrase "the plane whose equation is $E_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$"; this usage does not frequently lead to confusion.

the expression "pencil of planes" to designate the set of all the planes which pass through a line, we obtain the following theorem.

THEOREM 18. **The pencil of planes through the line of intersection of the planes E_1 and E_2 is represented by the equation $k_1E_1 + k_2E_2 = 0$, in which k_1 and k_2 are arbitrary constants not both zero.**

Remark 1. Since the geometrical significance of equation (3) is not altered when it is multiplied through by a non-zero constant, the pencil of planes through the intersection of the planes E_1 and E_2 is also represented by the equation $E_1 + kE_2 = 0$, where $k = \frac{k_2}{k_1}$, except that in the latter form the plane E_2 which is obtained from equation (3) when $k_1 = 0$, is excluded. For this reason the form (3) of the equation of the pencil deserves preference.

Remark 2. The lack of definiteness in the equations of a line pointed out in the remark at the beginning of Section 47 (page 83) can now, at least partially, be provided for, inasmuch as we can say that the line which is given by the pair of linear equations $E_1 = 0$ and $E_2 = 0$ is also determined by any two equations of the form (3), that is, by any two planes of the pencil of planes through this line.

Remark 3. The ratio $k_1 : k_2$ is a parameter in the equation (3) of the pencil of planes. It is constant for any one plane of the pencil, it varies as we pass from one plane in the pencil to another. The pencil of planes is a "one parameter family of planes." It will be instructive for the reader to compare the character of the parameter $k_1 : k_2$ in equation (3) with that of the parameters r, s, and t in the parametric equations of the line (see Section 47, page 86).

Remark 4. The method used for determining the equation of the pencil of planes finds frequent application throughout Analytical Geometry (see e. g. Exercise 3, Section 73, page 148 and Section 82, page 168). In connection with one of the remarks made in the opening paragraph of the present section, we observe that equation (3) does not represent a plane, if k_1 and k_2 are not both constants. The surface which it does represent will, however, still pass through the line of intersection of the planes E_1 and E_2. Furthermore if $S_1 = 0$ and $S_2 = 0$ are the equations of two arbitrary surfaces, the equation $k_1S_1 + k_2S_2 = 0$ represents a

surface which passes through all the points common to the two given surfaces, no matter what k_1 and k_2 may be.

Examples.

1. To determine the equation of a plane which passes through the line of intersection of the planes $3x - 2y + z - 4 = 0$ and $x + 5y - 2z + 3 = 0$ and which is moreover perpendicular to the plane $2x + y - 3z + 1 = 0$, we consider the pencil of planes through the given line. The equation of this pencil is

$$k_1(3x - 2y + z - 4) + k_2(x + 5y - 2z + 3) = 0.$$

If a plane of this pencil is to be perpendicular to the plane $2x + y - 3z + 1 = 0$, k_1 and k_2 must satisfy the condition which follows from the Corollary of Theorem 9 (Section 46, p. 82), namely, $(3k_1 + k_2)2 + (-2k_1 + 5k_2) + (k_1 - 2k_2)(-3) = 0$. This leads to the condition $k_1 + 13k_2 = 0$, that is, $k_1 : k_2 = 13 : -1$. The equation of the required plane is therefore $13(3x - 2y + z - 4) - (x + 5y - 2z + 3) = 0$, or $38x - 31y + 15z - 55 = 0$.

2. To determine a plane through the line whose parametric equations are

$$x = -4 + 3t, \quad y = 5 - t, \quad z = 3 + 2t$$

and through the point $P(-4, 3, 3)$.

First solution. The parameter t may be eliminated between the first two of the parametric equations of the line and also between the last two. This furnishes the two linear equations $x + 3y - 11 = 0$ and $2y + z - 13 = 0$; and the given line is the line of intersection of the planes which these equations represent. The equation of the pencil of planes through the given line can therefore be written in the form $k_1(x + 3y - 11) + k_2(2y + z - 13) = 0$. Since the point $P(-4, 3, 3)$ must lie on the required plane, the constants k_1 and k_2 must be so selected that $k_1(-4 + 9 - 11) + k_2(6 + 3 - 13) = 0$, or so that $-6k_1 - 4k_2 = 0$; hence $k_1 : k_2 = 2 : -3$. We conclude that the equation of the required plane is $2(x + 3y - 11) - 3(2y + z - 13) = 0$ or $2x - 3z + 17 = 0$.

Second solution. The required plane is determined by P and any two points on the line. Such points can be found at once when the line is given by parametric equations, by assigning two values arbitrarily to the parameter. The values $t = 0$ and $t = 1$ yield the points $A(-4, 5, 3)$ and $B(-1, 4, 5)$. The equation of the plane can now be written in the three-point form (see Theorem 5, Section 41, page 73). Thus we find the equation

$$\begin{vmatrix} x & y & z & 1 \\ -4 & 3 & 3 & 1 \\ -4 & 5 & 3 & 1 \\ -1 & 4 & 5 & 1 \end{vmatrix} = 0,$$

which, upon development, reduces to the form $2x - 3z + 17 = 0$, found by the first method.

If the plane $ax + by + cz + d = 0$ is to pass through a fixed point $P(\alpha, \beta, \gamma)$, its coefficients must satisfy the condition $a\alpha + b\beta + c\gamma + d = 0$, so that $d = -a\alpha - b\beta - c\gamma$; and it should be clear that if d has this value, the point P will lie on the plane. Upon introduction of the term "bundle of planes" to designate the set of planes which pass through a fixed point, we can state the following theorem.

THEOREM 19. **The bundle of planes through the point $P(\alpha, \beta, \gamma)$ is represented by the equation $a(x - \alpha) + b(y - \beta) + c(z - \gamma) = 0$, in which a, b, and c are arbitrary constants, not all zero.**

Remark. The ratios of the constants a, b, and c are the parameters in the equation of the bundle of planes; the bundle of planes is a "two parameter family of planes."

50. Exercises.

1. Determine the equation of a plane through the line of intersection of the planes $3x - y + 2z + 2 = 0$ and $2x + 4y - 3z + 1 = 0$, and

(*a*) through the point $A(-1, 3, -2)$;
(*b*) perpendicular to the plane $4x - 5y + z - 2 = 0$;
(*c*) through the origin;
(*d*) parallel to the Y-axis;
(*e*) parallel to the Z-axis.

2. Determine the equation of a plane through the line $x = 2 - 3t$, $y = 1 + 6t$, $z = -3 - 2t$ and through the line $x = 2 + t$, $y = 1 - 2t$, $z = -3 + 2t$.

3. Write the equation of a plane through the point $A(-3, 4, -1)$ and perpendicular to the line $\dfrac{x+2}{-3} = y - 2 = \dfrac{z-4}{2}$.

4. Prove analytically that every pencil of planes contains at least one plane parallel to the X-axis, at least one parallel to the Y-axis and at least one parallel to the Z-axis. Under what conditions will a pencil contain more than one plane in such position?

5. Prove that every bundle of planes contains exactly one plane parallel to the YZ-plane, one parallel to the ZX-plane and one parallel to the XY-plane.

6. Determine the equation of the plane through the line of intersection of the planes $3x - 6y - 2z + 5 = 0$ and $2x - y - 2z + 3 = 0$ which is perpendicular to the first of these planes.

7. Find the equations of the planes which bisect the angles between the planes $2x - 6y - 3z + 1 = 0$ and $4x + y - 8z + 5 = 0$. *Hint:* This problem can be solved by observing that the bisecting planes belong to the pencil of planes through the line of intersection of the given planes. Another

method of procedure is based on regarding the bisecting planes as the locus of points whose distances from the two given planes are equal or equal numerically but opposite in sign.

8. Find the equations of the planes which bisect the angles between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

51. Three Planes. A Plane and a Line. In this section we shall be concerned with the question of determining from the equations of three planes, E_1, E_2, and E_3, how they are placed with respect to each other, that is, with the problem of extending the result stated for two planes in Theorem 8 (Section 46, page 81). Let the equations of the planes be

$$E_1 \equiv a_1x + b_1y + c_1z + d_1 = 0, \quad E_2 \equiv a_2x + b_2y + c_2z + d_2 = 0,$$
$$E_3 \equiv a_3x + b_3y + c_3z + d_3 = 0.$$

It should be clear that if no two of the planes are parallel or coincident, the ranks of the c.m. and of the a.m. of the system of equations must be at least 2 (compare Theorem 8, Section 46, page 81). We obtain further results by means of Theorems 1 and 8 of Chapter II (see Sections 21, page 36, and 27, page 44). If the coefficient matrix is of rank 3, the system of equations has a unique solution; in this case the three planes have a single point in common. If the coefficient matrix has rank 2, the system of equations possesses a single infinitude of solutions or no solution, according as the rank of the a.m. is 2 or 3; in this case therefore the planes will have a line in common or no point in common, according as the rank of the augmented matrix is 2 or 3. If the rank of the c.m. is 1, the rank of the a.m. can not exceed 2 (Why?). In case it is 2, at least one pair of planes must be parallel; if it is 1, the three planes must be coincident. In this case therefore the planes are coincident or else they have no point in common, according as the rank of the a.m. is 1 or 2. We have therefore obtained the following conclusion.

Theorem 20. **Three planes will (1) have a single point in common if and only if the rank of the coefficient matrix of its equations is 3; (2) have a single line in common if and only if the ranks of the coefficient matrix and the augmented matrix are both 2; (3) be coincident if and only if the ranks of the coefficient matrix and the augmented matrix are both equal to 1; (4) have no point in common if and only if the ranks of the coefficient matrix and the augmented matrix are unequal.**

Remark 1. If the planes have a single point in common, they form a trihedral angle, see Fig. 10*a*; if they have a single line in common, they are three planes of a pencil (see Fig. 10*b*), unless two of the planes coincide; if they have no points in common, they form a triangular prism (see Fig. 10*c*), unless there is a pair of parallel or coincident planes among them. Since parallelism and coincidence of planes are readily determined by inspection of their equations (compare Exercise 1, Section 48, page 89), the following corollary of Theorem 20 is of considerable use in numerical cases.

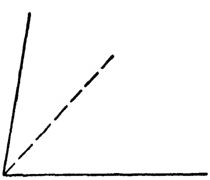

FIG. 10*a*

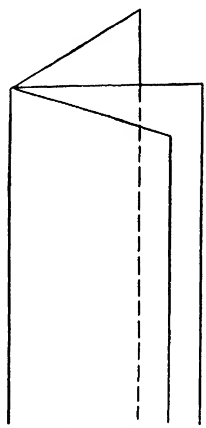

FIG. 10*b*

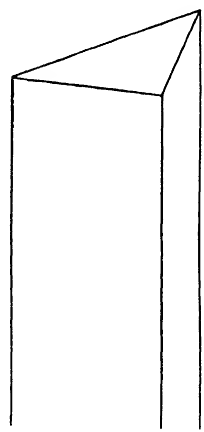

FIG. 10*c*

COROLLARY 1. **Three planes of which no two are either parallel or coincident will form (1) a trihedral angle if and only if the rank of the coefficient matrix of their equations is 3; (2) a pencil of planes if and only if the ranks of the coefficient matrix and the augmented matrix are both equal to 2; (3) a triangular prism if and only if the rank of the coefficient matrix is 2, while the rank of the augmented matrix is 3.**

For future reference it is convenient to state separately the following immediate deduction from Theorem 20.

COROLLARY 2. **Three planes have one or more points in common if and only if the ranks of the augmented matrix and of the coefficient matrix of their equations are equal.**

If the rank of the c.m. is not less than 2, two of the three equations can be taken as the equations of a line. The corresponding restatement of Theorem 20 leads to the following Corollary.

Corollary 3. **A plane and a line will (1) meet in a point if and only if the rank of the coefficient matrix of the three equations used to represent them is 3; (2) be parallel if and only if the ranks of the coefficient matrix and the augmented matrix of these equations are 2 and 3 respectively. The line will lie in the plane if and only if the rank of each of these matrices is 2.**

Remark. It is of interest to observe that these conditions must continue to hold true when the equations of the line are replaced by the equations of any two planes of the pencil of planes through the line. Hence the rank of the matrices of the 3 linear functions E_1, E_2, E_3 is not changed if the functions E_1 and E_2 are replaced by $k_1E_1 + k_2E_2$ and $l_1E_1 + l_2E_2$ respectively. Thus we obtain, for the special case $n = 3$, a geometrical interpretation of a part at least of Theorem 14, Chapter I (Section 10, page 18).

52. The Plane and the Line, *continued.* The geometrical content of Corollary 3 in the preceding section will become more apparent if the conditions of that corollary are interpreted in terms of the direction cosines of the line. Let the equations of the line be given in the form stated in Theorem 12 (Section 47, page 84):

$$\frac{x-\alpha}{\lambda} = \frac{y-\beta}{\mu} = \frac{z-\gamma}{\nu}.$$

It will always be possible to select among the three equations here represented two whose c.m. has rank 2; these equations can then be taken as the equations of the line. If we suppose that $\mu \neq 0$, these equations may be taken to be the following two:

$$\mu x - \lambda y - \alpha\mu + \beta\lambda = 0 \quad \text{and} \quad \nu y - \mu z - \beta\nu + \gamma\mu = 0.$$

The condition that the given line shall meet the plane $ax + by + cz + d = 0$ in a point, can therefore, in virtue of Corollary 3, Section 51, (1), be written in the form:

$$(1) \quad \begin{vmatrix} \mu & -\lambda & 0 \\ 0 & \nu & -\mu \\ a & b & c \end{vmatrix} \neq 0.$$

If we develop this determinant and divide out the factor μ which was supposed to be different from zero, we find the condition

$$(2) \quad a\lambda + b\mu + c\nu \neq 0.$$

On the other hand the line will lie in the plane or be parallel to it if and only if the determinant in (1) vanishes, that is, since $\mu \neq 0$, if and only if

$$(3) \quad a\lambda + b\mu + c\nu = 0$$

while at least one two-rowed minor of the determinant in (1) has a value different from zero.

To distinguish the case of parallelism from the case in which the line lies in the plane, we have to consider the augmented matrix of the system of equations

$$(4) \quad \begin{gathered} \mu x - \lambda y - \alpha\mu + \beta\lambda = 0, \quad \nu y - \mu z - \beta\nu + \gamma\mu = 0, \\ ax + by + cz + d = 0. \end{gathered}$$

There is no loss of generality, by virtue of the hypothesis that the rank of the matrix of the determinant in (1) is 2, if we suppose that not all the cofactors of the elements in the second column of this determinant vanish; let us denote the values of these cofactors by C_1, C_2, and C_3 and let us suppose that $C_1 \neq 0$. We know then from Theorem 14, Chapter I (Section 10, page 18), that the rank of the augmented matrix of the system of equations (4) is not changed if the first row is replaced by C_1 times the first row, plus C_2 times the second row plus C_3 times the third row. But if this operation is carried out, the first and third elements of this row will reduce to zero by Theorem 13, Chapter I (Section 7, page 13), the second element vanishes on account of (3), and the fourth element becomes

$$C_1(-\alpha\mu + \beta\lambda) + C_2(-\beta\nu + \gamma\mu) + C_3 d.$$

Consequently the augmented matrix of the system of equations (4) will have rank 2 or 3 according as this last expression is or is not equal to zero. Now, $C_1 = -a\mu$, $C_2 = c\mu$ and $C_3 = \mu^2$. If these values are substituted in the expression for the fourth element, above, we find that the rank of the augmented matrix is 2 or 3 according as the equation

$$a\mu(\alpha\mu - \beta\lambda) + c\mu(-\beta\nu + \gamma\mu) + \mu^2 d = 0$$

is or is not satisfied. This condition reduces to

$$(a\alpha + c\gamma + d)\mu^2 - (a\lambda + c\nu)\beta\mu = 0.$$

But from (3) it follows that $a\lambda + c\nu = -b\mu$; if this is used in the preceding equation and if the non-vanishing factor μ^2 is divided out, we are led to the condition

$$(5) \quad a\alpha + b\beta + c\gamma + d = 0.$$

We have therefore obtained the following equivalent form of Corollary 3 of Theorem 20.

THEOREM 21. **The line l through the point $P(\alpha, \beta, \gamma)$ and with direction cosines λ, μ, ν will (1) meet the plane $ax + by + cz + d = 0$ in a single point if and only if $a\lambda + b\mu + c\nu \neq 0$; (2) be parallel to the plane if and only if $a\lambda + b\mu + c\nu = 0$ and $a\alpha + b\beta + c\gamma + d \neq 0$; or lie in the plane if and only if $a\lambda + b\mu + c\nu = 0$ and $a\alpha + b\beta + c\gamma + d = 0$.**

Remark 1. The geometrical interpretation of these conditions should be obvious. For it follows from Corollary 3 of Theorem 7 (Section 44, page 79) that the condition (3) requires that the angle between the line and the plane shall be 0° or 180°, that is, that the line shall lie in the plane or be parallel to it. And the condition (5) clearly states that the point $P(\alpha, \beta, \gamma)$ must lie in the plane $ax + by + cz + d = 0$.

Remark 2. We have been interested in deriving Theorem 21 from Theorem 20 in order to illustrate the power of this general theorem. It must be observed, however, that the conditions of Theorem 21 are obtained more directly if the equations of the line are taken in the parametric form of Theorem 15 (Section 47, page 86), $x = \alpha + \lambda s$, $y = \beta + \mu s$, $z = \gamma + \nu s$. For if these expressions are substituted for x, y, and z in the equation of the plane, $ax + by + cz + d = 0$, we obtain the linear equation:

$$(a\lambda + b\mu + c\nu)s + (a\alpha + b\beta + c\gamma + d) = 0$$

from which the value of s is to be ascertained, which determines the point of intersection of the line with the plane. From this equation it is evident that, if $a\lambda + b\mu + c\nu \neq 0$, the equation has a single root and the line meets the plane in a single point; if $a\lambda + b\mu + c\nu = 0$ and $a\alpha + b\beta + c\gamma + d \neq 0$, the equation has no solution, and the line is parallel to the plane; if $a\lambda + b\mu + c\nu = 0$ and $a\alpha + b\beta + c\gamma + d = 0$, the equation is satisfied by every value of s and the line lies in the plane.

Examples.

1. Let it be required to determine the equation of a plane through the point $A(-4, 1, 2)$ and parallel to the lines

$$l_1 : \frac{x-2}{4} = \frac{y+1}{-2} = \frac{z-3}{3} \quad \text{and} \quad l_2 : \frac{x+3}{-1} = \frac{y-4}{3} = \frac{z+2}{-2}.$$

If $P(x, y, z)$ is an arbitrary point on the required plane, it must be possible to determine four numbers a, b, c, and d, not all zero, such that

$$\begin{aligned} ax + by \quad + cz \quad + d &= 0; \text{ such that} \\ -4a + b \quad + 2c + d &= 0, \text{ since } A \text{ is to lie in the plane; and such that} \\ 4a - 2b + 3c \qquad &= 0, \text{ and} \\ -a + 3b \;\; - 2c \qquad &= 0, \text{ since } l_1 \text{ and } l_2 \text{ are to be parallel to the plane} \end{aligned}$$

and their direction cosines are proportional to 4, -2, 3 and to -1, 3, -2 respectively. These four linear homogeneous equations in a, b, c, and d possess a non-trivial solution only if the value of the coefficient determinant is zero (see Corollary of Theorem 2, Chapter II, Section 22, page 38). Hence the condition on the coördinates of P is that $\begin{vmatrix} x & y & z & 1 \\ -4 & 1 & 2 & 1 \\ 4 & -2 & 3 & 0 \\ -1 & 3 & -2 & 0 \end{vmatrix} = 0$, or that $x - y - 2z + 9 = 0$. The locus of this linear equation is a plane; it is easy to verify that this plane meets the required conditions.

2. If it is required to show that the line of intersection of the planes $x - 2y + z - 4 = 0$ and $3x + 5y - 2z + 4 = 0$ is parallel to the plane $7x - 3y + 2z - 5 = 0$, we can proceed in various ways. Using Corollary 3 of Theorem 20 (Section 51, page 97), we can show that $\begin{vmatrix} 1 & -2 & 1 \\ 3 & 5 & -2 \\ 7 & -3 & 2 \end{vmatrix} = 0$ and that the rank of the matrix $\left\| \begin{matrix} 1 & -2 & 1 & -4 \\ 3 & 5 & -2 & 4 \\ 7 & -3 & 2 & -5 \end{matrix} \right\|$ is 3; thus the problem is made to depend on the evaluation of determinants. We can also reduce the equations of the line to the point-direction form, established in Theorem 12 (Section 47, page 84) and then apply Theorem 21.

53. Exercises.

1. Determine the relative positions of the planes in each of the following sets:

(*a*) $3x - 2y + 4z = 0,\ 2x + 3y - z + 3 = 0,\ x - 4y + 2z + 2 = 0;$
(*b*) $x - 2y - 4z + 3 = 0,\ 3x + y - z + 2 = 0,\ 3x + 8y + 10z - 5 = 0;$
(*c*) $2x + y - 3z + 4 = 0,\ 4x + 2y - 6z + 5 = 0,\ 3x - 6y + z + 1 = 0;$
(*d*) $x - y + z - 3 = 0,\ 2x + y + 3z = 0,\ 3x + 5y + 3z - 1 = 0;$
(*e*) $4x + y - 3z + 2 = 0,\ 2x - 3y + z - 4 = 0,\ 7y - 5z + 4 = 0;$
(*f*) $x + y + z + 1 = 0,\ 2x + 2y + 3z - 4 = 0,\ 2x + 2y + 2z + 1 = 0;$
(*g*) $5x - 2y - 7z + 3 = 0,\ 10x - 4y - 14z - 2 = 0,\ 15x - 6y - 21z - 8 = 0.$

2. For those sets of planes in the preceding exercise which form a trihedral angle, determine the point common to the three planes.

3. For those sets of planes in Exercise 1 which belong to a pencil of planes, determine the direction cosines of the line common to the three planes.

4. Set up the equation of a plane through the point $A(2, -1, 3)$ and parallel to the lines $x = 2 - 3t$, $y = -1 + 2t$, $z = 3 - t$ and $x = 4 + 3t$, $y = -3 + 5t$, $z = 1 - 2t$.

5. Set up the equation of a plane through the point $P(\alpha, \beta, \gamma)$ and parallel to the lines $x = \alpha_i + \lambda_i s, \quad y = \beta_i + \mu_i s, \quad z = \gamma_i + \nu_i s, \quad i = 1, 2.$

6. Prove that the ranks of the coefficient matrix and the augmented matrix of the equations of three planes are 2, if these planes belong to the same pencil of planes; in proving, use the results of Section 49 and Theorem 14, Chapter I.

54. Four Planes. Two Lines. The number of possible relative positions of a set of planes increases quite rapidly when the set contains more than three planes. The methods to be used in such cases do not differ in any essential respect, however, from those employed in the preceding sections. We shall not study this general problem therefore; we shall restrict ourselves to the following special cases.

Theorem 22. **Four planes meet in a single point if and only if the ranks of the coefficient matrix and the augmented matrix of their equations are both 3.**

Proof. If the planes meet in a single point there must be at least one set of three among them which form a trihedral angle; in that case at least one of the three-rowed minors of the c.m. has a value different from zero and therefore the rank of the c.m. is 3. Moreover, it follows from Theorem 3, Chapter II (Section 23, page 38) that the determinant of the augmented matrix vanishes; hence the rank of that matrix is less than 4. But since the rank of the a.m. can certainly not be less than that of the c.m., the rank of the a.m. is 3.

Conversely, if the ranks of the c.m. and the a.m. are both 3, we conclude by means of Theorem 3, Chapter II, that the equations have a unique solution, that is, that the planes have a single point in common.

Corollary. **Two lines meet in a point if and only if the coefficient matrix and the augmented matrix of the four linear equations used to represent them, both have rank 3.**

Theorem 23. **Four planes have a single line in common if and only if the ranks of the coefficient matrix and the augmented matrix of their equations are both equal to 2.**

Proof. If the four planes have a single line in common then every set of three of them have at least a line in common, and there must be at least one set of three which have but a single line in common. It follows therefore from Theorem 20 (Section 51, page 95) that the ranks of the c.m. and the a.m. of any three of the four equations is at most 2, and that there is one set of three equations among them at least, whose c.m. and a.m. both have rank 2. We conclude from this that the ranks of the c.m. and the a.m. of the four equations must both be 2.

Conversely, if the rank of both these matrices is 2, then for any three of the four equations the ranks of both the c.m. and the a.m. are at most 2, whereas there is at least one set of three equations for which the ranks of the c.m. and the a.m. are exactly 2. It follows therefore from Theorem 20 that there is at least one set of three among the four planes which meet in a line, while the fourth plane either passes through this same line, or else is parallel to it. The latter alternative is ruled out by Corollary 3 of Theorem 20, Section 51, page 97. Hence the fourth plane passes through the line common to the other three, as is required by the theorem.

THEOREM 24. **Four planes coincide if and only if the ranks of the coefficient matrix and the augmented matrix of their equations are 1.**

The proof is left to the reader.

From Theorems 22, 23, and 24 we obtain immediately the following corollaries:

COROLLARY 1. **Four planes have one or more points in common if and only if the ranks of the coefficient matrix and the augmented matrix of their equations are equal.**

COROLLARY 2. **Four planes have no points in common if the augmented matrix of its equations is non-singular.** (Compare Definition III, Chapter II, Section 26, p. 43.)

We return now to the Corollary of Theorem 22. The criterion for deciding whether two lines meet in a point, which it supplies, is not very convenient if the parametric forms of the equations of the lines are used. We shall therefore develop this criterion in a different form.

Let us consider the lines

$$l_1 : x = \alpha_1 + \lambda_1 s, \quad y = \beta_1 + \mu_1 s, \quad z = \gamma_1 + \nu_1 s,$$

and

$$l_2 : x = \alpha_2 + \lambda_2 s', \quad y = \beta_2 + \mu_2 s', \quad z = \gamma_2 + \nu_2 s'.$$

Whether or not these lines have one or more points in common depends upon whether or not it is possible to determine one or more values of s and s' such that

$$\alpha_1 + \lambda_1 s = \alpha_2 + \lambda_2 s', \quad \beta_1 + \mu_1 s = \beta_2 + \mu_2 s', \quad \gamma_1 + \nu_1 s = \gamma_2 + \nu_2 s';$$

that is, upon whether or not the system of equations

$$\text{(1)} \quad \lambda_1 s - \lambda_2 s' + \alpha_1 - \alpha_2 = 0, \quad \mu_1 s - \mu_2 s' + \beta_1 - \beta_2 = 0, \\ \nu_1 s - \nu_2 s' + \gamma_1 - \gamma_2 = 0$$

possesses solutions.

The c.m. of this system of equations is

$$\text{(2)} \quad \left\| \begin{matrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{matrix} \right\|;$$

its rank is 2, unless the lines are parallel or coincident. We conclude therefore, on the basis of Theorem 3, Chapter II (Section 23, page 38), that if the lines are neither parallel nor coincident they will have one point in common or none according as the rank of the augmented matrix of the system (1), that is, of the matrix

$$\text{(3)} \quad \left\| \begin{matrix} \lambda_1 & \lambda_2 & \alpha_1 - \alpha_2 \\ \mu_1 & \mu_2 & \beta_1 - \beta_2 \\ \nu_1 & \nu_2 & \gamma_1 - \gamma_2 \end{matrix} \right\|$$

is 2 or 3. If two non-parallel lines have no point in common, they are said to lie skew with respect to each other.

If the rank of the matrix (2) is 1, the lines are parallel or coincident; and to distinguish between them we have to consider the rank of the matrix (3). If this is 1, all its rows are proportional, so that if any one of the equations is satisfied, the other two will also be satisfied. This means that to any value of s there corresponds a value of s' such that together they will satisfy the equations (1). Hence every point on l_1 coincides with some point on l_2; in other words, the lines l_1 and l_2 coincide. If the rank of the matrix (3) is 2, there is at least one pair among the equations which do not possess a solution, in virtue of Theorem 3, Chapter

II; in this case therefore the lines can have no point in common. Finally, it is clear that the rank of (3) can not be 3 if the rank of (2) is 1; for in that case all the cofactors of the elements in the last column of (3) vanish.

We summarize the conclusions in a theorem.

THEOREM 25. **The two lines $x = \alpha_i + \lambda_i s$, $y = \beta_i + \mu_i s$, $z = \gamma_i + \nu_i s$ are skew if and only if the rank of the matrix (3) is 3; they meet in a point if and only if the ranks of the matrices (2) and (3) are both 2; they are parallel if and only if the rank of the matrix (2) is 1, while the rank of the matrix (3) is 2; they are coincident if and only if the ranks of the matrices (2) and (3) are both 1.**

Remark. It should be clear that similar conclusions are obtained if the equations of the lines are taken in the parametric form of Theorem 16 (Section 47, page 86).

Examples.

1. To find the relative position of the lines l_1, given by the equations $2x - y + 3z + 4 = 0$, $x + 2y - z - 3 = 0$, and l_2, given by the equations $3x + 2y - 2z + 5 = 0$, $2x - 3y + z - 4 = 0$, we begin by finding the direction cosines of each. We find, by use of Theorem 17 (Section 47, page 87) that

$$\lambda_1 : \mu_1 : \nu_1 = \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} : - \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} : \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = -1 : 1 : 1, \text{ and}$$

$$\lambda_2 : \mu_2 : \nu_2 = \begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix} : - \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} : \begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix} = 4 : 7 : 13.$$

It is evident from these results that the lines are neither parallel nor coincident. To decide whether or not they are skew, we determine a point on each of the lines. The point $P_1(-2, 3, 1)$ lies on l_1; the point $P_2(1, 1, 5)$ lies on l_2. And the determinant

$$\begin{vmatrix} -1 & 4 & -3 \\ 1 & 7 & 2 \\ 1 & 13 & -4 \end{vmatrix} = \begin{vmatrix} -1 & 4 & -3 \\ 0 & 11 & -1 \\ 0 & 17 & -7 \end{vmatrix} \neq 0.$$

Hence the lines are skew.

2. The lines $l_1 : x = -4 + t$, $y = 3 - 2t$, $z = 2 + 3t$ and $l_2 : x = -2 + 3t'$, $y = -1 - 6t'$, $z = 8 + 9t'$ are parallel or coincident, since their direction cosines are proportional to each other. The augmented matrix of the equations $t - 3t' - 2 = 0$, $-2t + 6t' + 4 = 0$ and $3t - 9t' - 6 = 0$ is $\begin{Vmatrix} 1 & -3 & -2 \\ -2 & 6 & 4 \\ 3 & -9 & -6 \end{Vmatrix}$; and we see by inspection that its rank is 1. Hence the two lines coincide; the substitution $t = 3t' + 2$ carries the equations of l_1 over into those of l_2.

55. Exercises.

1. Show that the four planes $3x + y - z - 5 = 0$, $x - 2y + 3z - 2 = 0$, $2x + 4y - 5z - 2 = 0$, and $-4x + 3y - 7z + 3 = 0$ meet in a point. Determine the coördinates of this point.

2. Show that the four planes $3x - y + 2z - 3 = 0$, $2x + 2y - 3z + 4 = 0$, $x + 5y - 8z + 11 = 0$ and $8y - 13z + 18 = 0$ meet in a line. Determine the direction cosines of this line.

3. Show that the lines $x = 4 - 2t$, $y = -3 + 2t$, $z = 5 - 3t$ and $x = t$, $y = 1 - 4t$, $z = -1 + 3t$ meet in a point. Determine the coördinates of this point.

4. Determine whether the lines $x = 5 - 3t$, $y = 4 + t$, $z = -3 + 4t$ and $x = 6 - 6t$, $y = -2 + 2t$, $z = 5 + 8t$ are parallel or coincident.

5. Prove that if four planes form a tetrahedron, the ranks of the coefficient matrix and the augmented matrix of their equations are 3 and 4 respectively.

6. Prove that if four planes form a four-sided prism, the ranks of the coefficient matrix and the augmented matrix of their equations are 2 and 3 respectively.

7. Prove analytically that if two lines are parallel there exists one and only one plane in which they both lie.

8. Remembering that skew lines are non-parallel lines which have no point in common, prove that if two lines are skew, there exists no plane in which they both lie.

56. Miscellaneous Exercises.

1. Determine the distance of the point $A(-2, 3, 8)$ from the line $l : x = 3 - 2t$, $y = 1 + 2t$, $z = -6 - t$. *Hint:* The distance from the line to A is equal to the product of the distance from A to the point $B(3, 1, -6)$ on the line by the sine of the angle between AB and l; use Theorem 14, Chapter III, (Section 36, page 64).

2. Determine the distance of the point $P(\alpha_1, \beta_1, \gamma_1)$ from the line $x = \alpha + \lambda s$, $y = \beta + \mu s$, $z = \gamma + \nu s$.

3. Find the distances of the point $A(4, -5, -3)$ from the planes $2x - 6y - 3z + 4 = 0$ and $3x + 6y - 2z - 5 = 0$ and from their line of intersection.

4. Determine the relative positions of the planes in the following sets:

(*a*) $2x - y + 3z - 4 = 0$, $3x + 2y - z + 2 = 0$, $x - 4y + 7z - 10 = 0$;
(*b*) $2x - y + 3z - 4 = 0$, $3x + 2y - z + 2 = 0$, $x - 4y + 7z - 6 = 0$.

5. Find the points which

(*a*) the plane $3x - 2y + z + 2 = 0$ and the line $x = -1 + t$, $y = -2 + 2t$, $z = -3 + 4t$;
(*b*) the plane $3x - 2y + z + 2 = 0$ and the line $x = -1 + 2t$, $y = 2 + 2t$, $z = 3 - 2t$;
(*c*) the plane $3x - 2y + z + 2 = 0$ and the line $x = -1 + t$, $y = -2 + t$, $z = -3 - t$

have in common.

6. The intercepts of a plane are a, b, and c. On the axes of another rectangular reference frame with the same origin as the original axes, the intercepts of the same plane are a_1, b_1, and c_1. Show that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a_1{}^2} + \frac{1}{b_1{}^2} + \frac{1}{c_1{}^2}$.

7. Show that the locus of all points which are equally distant from two given planes consists of two planes through the line of intersection of the given planes. For the points on one of these planes the distances from the two given planes are equal in sign as well as in magnitude; for those on the other plane the distances from the two given planes are equal in magnitude, but opposite in sign.

8. Prove that the two planes determined in Exercise 7 are perpendicular to each other.

9. Prove that each of the planes determined in Exercise 7 makes equal angles with the two given planes. The planes found in Exercise 7 are called the *bisecting planes* of the dihedral angle formed by the two given planes.

10. Prove that if three planes meet in a point, the six bisecting planes of the three dihedral angles formed by them meet three by three in a line. *Hint:* Take the equations of the given planes in the normal form.

11. Write the equation of the plane through the origin determined by the two lines through the origin whose direction cosines are λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2.

12. Show that three concurrent lines are coplanar (lie in one plane) if and only if the determinant formed by their direction cosines vanishes. (Compare Exercise 5, Section 42, page 74.)

> *Note.* The determinant, mentioned in this exercise, whose rows consist of the direction cosines of three concurrent lines, will be called the *orientation determinant* of these lines.

13. A point moves in such a manner that its distances from two fixed lines are always equal to each other. Determine the equation of the locus which this point describes.

14. Determine the equations of the line which passes through the point $P_1(\alpha_1, \beta_1, \gamma_1)$, is perpendicular to the line joining $P_2(\alpha_2, \beta_2, \gamma_2)$ and $P_3(\alpha_3, \beta_3, \gamma_3)$, and lies in the plane determined by the points P_1, P_2, and P_3.

15. Determine the distance of the point $P_1(\alpha_1, \beta_1, \gamma_1)$ from the line joining the points $P_2(\alpha_2, \beta_2, \gamma_2)$ and $P_3(\alpha_3, \beta_3, \gamma_3)$.

16. Show that four times the square of the area of the triangle whose vertices are $P_i(\alpha_i, \beta_i, \gamma_i)$, $i = 1, 2, 3$ is equal to the sum of the squares of the determinants $\begin{vmatrix} \beta_1 & \gamma_1 & 1 \\ \beta_2 & \gamma_2 & 1 \\ \beta_3 & \gamma_3 & 1 \end{vmatrix}$, $\begin{vmatrix} \gamma_1 & \alpha_1 & 1 \\ \gamma_2 & \alpha_2 & 1 \\ \gamma_3 & \alpha_3 & 1 \end{vmatrix}$, $\begin{vmatrix} \alpha_1 & \beta_1 & 1 \\ \alpha_2 & \beta_2 & 1 \\ \alpha_3 & \beta_3 & 1 \end{vmatrix}$.

17. Find the distance of the point $P(\alpha, \beta, \gamma)$ from the plane determined by the points $P_i(\alpha_i, \beta_i, \gamma_i)$, $i = 1, 2, 3$.

18. Prove that the volume of the tetrahedron whose vertices are the points $P_i(\alpha_i, \beta_i, \gamma_i)$ $i = 1, 2, 3, 4$ is equal to one sixth of the value of the determinant

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & 1 \\ \alpha_2 & \beta_2 & \gamma_2 & 1 \\ \alpha_3 & \beta_3 & \gamma_3 & 1 \\ \alpha_4 & \beta_4 & \gamma_4 & 1 \end{vmatrix}.$$

19. Determine the equations of the three planes, each of which passes through one of three concurrent edges of a tetrahedron and is perpendicular to the face opposite to the vertex in which these edges meet.

20. Show that the three planes determined in the preceding exercise meet in a line which passes through one vertex and is perpendicular to the opposite face.

21. Prove that the four perpendiculars from the vertices of a tetrahedron to the opposite faces meet in a point.

22. Determine the equation of the plane through one edge of a tetrahedron and through the midpoint of the opposite edge.

23. Prove that the six planes of the kind described in the preceding exercise meet in a point.

24. Determine the equation of the plane through the midpoint of one edge of a tetrahedron and perpendicular to the opposite edge.

25. Prove that the six planes of the kind described in the preceding exercise have one point in common.

CHAPTER V

OTHER COÖRDINATE SYSTEMS

In the development of the subject up to this point we have used a rectangular Cartesian coördinate system; and we have had no occasion to change from one such system to another. Many of the problems to be taken up in later chapters require such transitions; and for other purposes it is frequently desirable to use reference frames different from that furnished by the rectangular Cartesian coördinates. We shall therefore consider in the present chapter some other reference frames in three-space, and also the transition from one reference frame to another.

57. Spherical Coördinates. The reference frame consists of: (1) a **fixed plane** Π; (2) a **fixed half-line,** l, in this plane, called the **initial line;** (3) a **fixed point** O on the line l, called the origin; and (4) a **unit for linear measurement** and a **unit for angular measurement.** To determine the coördinates of a point P in space with reference to this frame, we connect O with P and we drop a perpendicular from P to the plane Π; let P' be the foot of this perpendicular (see Fig. 11). The spherical coördinates of P are then defined as follows.

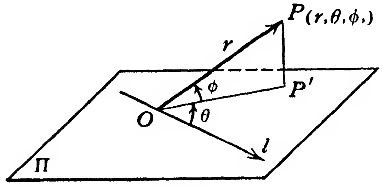

FIG. 11

DEFINITION I. **The spherical coördinates of a point P in space are: (1) the unsigned distance r from O to P, measured in terms of the unit specified for linear measurement — this is called the *radius vector* of P; (2) the angle ϕ between $-90°$ and $90°$ which the plane Π makes with OP, measured in terms of the unit specified for angular measurement — this is called the *latitude;* and (3) the angle θ between $0°$ and $360°$ which the line l makes with the projection of OP on the plane Π, measured in terms of the same unit — this is called the *longitude.***

Remark 1. It follows from this definition that to every point in space, *except* O, there corresponds a definite set of three real numbers, which are its spherical coördinates. But it is not true in this case, as it was when Cartesian coördinates were used, that

for every set of three real numbers there exists a point of which these numbers are the spherical coördinates. The radius vector is an unsigned real number, the latitude must lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and the longitude between 0 and 2π, if the radian is the unit of angular measurement (between -90 and 90, between 0 and 360 if the degree is the unit).

Remark 2. There are various ways in which the definition of spherical coördinates may be modified. The radius vector may be defined as a signed number, with a possibility of its being either positive or negative; and the ranges of value for the latitude and the longitude may be changed. Although there are some advantages to be derived from such different agreements which the coördinates, as defined above, do not possess, the present definition has the desirable property, mentioned in Remark 1, of assigning to every point in space, except O, a single set of spherical coördinates.

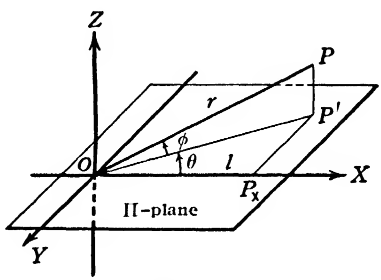

Fig. 12

To establish a connection between spherical coördinates and rectangular Cartesian coördinates, we make the point O the origin of a rectangular reference frame, the line l the positive X-axis and the plane Π the XY-plane. Moreover we adopt the unit of linear measurement as the unit on the three axes of this superimposed rectangular frame. It should now be easy to see (from Fig. 12) that the Cartesian coördinates of a point P are connected with its spherical coördinates by the following formulas:

$$\begin{aligned} x &= OP_x = OP' \cos\theta = r\cos\phi\cos\theta, \\ y &= P_xP' = OP' \sin\theta = r\cos\phi\sin\theta, \\ z &= P'P = r\sin\phi. \end{aligned}$$

If these equations are squared and then added together, we find, in accordance with the conventions laid down in Definition I, that $r = |\sqrt{x^2 + y^2 + z^2}|$ and hence that $\phi = \text{Arc sin}\, \dfrac{z}{|\sqrt{x^2 + y^2 + z^2}|}$.* Squaring and adding the first two equations leads to $r \cos \phi = |\sqrt{x^2 + y^2}|$, and hence to the result that $\sin \theta = \dfrac{y}{|\sqrt{x^2 + y^2}|}$ and $\cos \theta = \dfrac{x}{|\sqrt{x^2 + y^2}|}$; by means of these conditions the angle θ is completely determined between 0 and 2π. We state our results as follows.

THEOREM 1. **The transformation from spherical coördinates to rectangular Cartesian coördinates, and vice versa, when — in the two reference frames — the origins coincide, the initial line coincides with the positive half of the X-axis, the initial plane with the XY-plane, and the units of linear measurement are the same, is accomplished by means of the equations:**

$$x = r \cos \phi \cos \theta, \quad y = r \cos \phi \sin \theta, \quad z = r \sin \phi;$$

and

$$r = |\sqrt{x^2 + y^2 + z^2}|, \quad \phi = \text{Arc sin}\, \frac{z}{|\sqrt{x^2 + y^2 + z^2}|},$$

$$\sin \theta = \frac{y}{|\sqrt{x^2 + y^2}|}, \quad \cos \theta = \frac{x}{|\sqrt{x^2 + y^2}|}.$$

By means of the first set of formulas an equation in Cartesian coördinates may be transformed into an equation in spherical coördinates; and the second set of formulas enables us to transform an equation in spherical coördinates into an equation in Cartesian coördinates. In view of Definitions I and II of Chapter IV (Section 39, page 68), we conclude from this that the locus of a single equation in spherical coördinates is a surface, and the locus of a pair of equations a curve. The geometrically

* We are using here the notation Arc sin u to indicate the "principal value" of the angle whose sine is u, that is, the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ which has its sine equal to u; this function has a single real value for every real value of u between -1 and 1. It should be clear that $\dfrac{z}{|\sqrt{x^2 + y^2 + z^2}|}$ is never more than 1 and the angle ϕ as defined in the text always exists and lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

simplest surfaces, that is, the planes, are not represented by the algebraically simplest equation when spherical coördinates are used. By using Theorem 1, in connection with Theorem 4, Chapter IV (Section 41, page 71), we find that the general equation of a plane in spherical coördinates is

$$r(a \cos \phi \cos \theta + b \cos \phi \sin \theta + c \sin \phi) + d = 0.$$

On the other hand, the equation of a sphere whose center is at the origin and whose radius is a has, in spherical coördinates, the very simple equation $r = a$.

58. Cylindrical Coördinates. The reference frame now consists of the initial plane Π, the initial line l, the origin O, units of linear and of angular measurement, and besides of a directed perpendicular to the plane Π at O, called the $\mathfrak{Z}$-axis. The cylindrical coördinates of an arbitrary point P in space are then defined as follows (see Fig. 13).

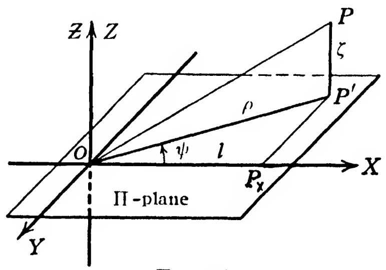

FIG. 13

DEFINITION II. **The *cylindrical coördinates* of a point P in space are: (1) the perpendicular distance, ζ, from the initial plane to P, measured in accordance with the unit of measurement and the direction specified for the $\mathfrak{Z}$-axis; (2) the undirected distance, ρ, from O to the projection P' of P on the initial plane, measured in terms of the specified unit of linear measurement; and (3) the angle ψ, between 0° and 360°, which the initial line l makes with OP', measured in terms of the specified unit of angular measurement.**

Remark. The cylindrical coördinates of a point evidently combine polar coördinates in the initial plane with a Cartesian ζ-coördinate. For every point in space, except those which lie on the $\mathfrak{Z}$-axis, there exists a unique set of 3 real numbers, which are its cylindrical coördinates. But again it is not true that with every set of three real numbers there is associated a point of which these numbers are the cylindrical coördinates. The ζ-coördinate is a signed real number, the coördinate ρ an unsigned real number, and the coördinate ψ is restricted to the range $0 - 2\pi$, if the radian is the unit of angular measurement.

A rectangular Cartesian reference frame can be superimposed on the reference frame used for cylindrical coördinates in the manner used in the preceding section for spherical coördinates. Thus we find the following relations between the rectangular Cartesian coördinates x, y, z of a point and its cylindrical coördinates ρ, ψ, ζ (see Fig. 13):

$$x = OP_x = \rho\cos\psi, \quad y = P_xP' = \rho\sin\psi, \quad z = \zeta;$$

and

$$\rho = |\sqrt{x^2+y^2}|, \quad \cos\psi = \frac{x}{|\sqrt{x^2+y^2}|}, \quad \sin\psi = \frac{y}{|\sqrt{x^2+y^2}|}, \quad \zeta = z.$$

Moreover, the reference frame for cylindrical coördinates contains a reference frame for spherical coördinates. It is therefore a simple matter to connect the spherical coördinates r, ϕ, θ of a point with its cylindrical coördinates ρ, ψ, ζ. We find:

$$\rho = OP' = r\cos\phi, \quad \psi = \theta, \quad \zeta = r\sin\phi;$$

and

$$r = |\sqrt{\rho^2+\zeta^2}|, \qquad \phi = \text{Arc}\tan\frac{\zeta}{\rho},^* \quad \theta = \psi.$$

A repetition of the argument made in the last paragraph of Section 57 should make it clear that the locus of a single equation in cylindrical coördinates is a surface, and the locus of a pair of equations a curve. The equation $\rho = a$ represents a right circular cylindrical surface whose radius is a and whose axis is along the $\mathfrak{Z}$-axis; the pair of equations $\rho = a$, $\zeta = b$ determines a circle of radius a, in a plane parallel to the initial plane at a distance b from it and having its center on the $\mathfrak{Z}$-axis.

59. Exercises.

1. Determine the loci of each of the following equations:

$(a)\ r = 2;\ (b)\ \rho = 3;\ (c)\ \theta = \frac{5\pi}{6};\ (d)\ \zeta = -1;\ (e)\ \phi = -\frac{\pi}{3};\ (f)\ \psi = \frac{5\pi}{4}.$

2. Write the equations in spherical coördinates of the surfaces whose equations in rectangular Cartesian coördinates are:

$(a)\ x^2 + y^2 = 5;\quad (b)\ y^2 + z^2 = 3;\quad (c)\ 3x - 2y = 0;$
$(d)\ 3x^2 + 2y^2 + 4z^2 = 1;\quad (e)\ 4x^2 - y^2 = 1.$

* The notation Arc tan u is used to designate the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is u, that is, the "principal value" of the multiple-valued function Arc tan u; see the footnote on page 110.

3. Determine the locus of each of the following pairs of equations:

$$(a)\ r = 3, \theta = \frac{7\pi}{6};\quad (b)\ \zeta = 4, \psi = \frac{\pi}{2};\quad (c)\ \phi = -\frac{\pi}{6}, \theta = \frac{2\pi}{3};$$
$$(d)\ \rho = 5, \zeta = -2;\quad (e)\ r = 4, \phi = \frac{\pi}{4}.$$

4. Transform the following equations into equations in rectangular Cartesian coördinates:

$$(a)\ r = \tan\theta;\quad (b)\ \zeta = 2\phi;\quad (c)\ r(\cos\theta + \sin\theta - \tan\phi) = 4\sec\phi;$$
$$(d)\ \rho(3\cos\psi - \sin\psi) + 2\zeta - 4 = 0;\quad (e)\ \rho^2 + \zeta^2 = 9;$$
$$(f)\ \sin^2\phi + 2\sin^2\theta - 3\cos^2\theta = 4.$$

60. Oblique Cartesian Coördinates. A reference frame for a system of oblique Cartesian coördinates is furnished by any three planes which meet in a point, O. These planes meet two by two, in lines through O; we call these lines the X-, Y-, and Z-axes and denote them by OX, OY, and OZ respectively (Fig. 14). On each axis we specify a positive direction and a unit of measurement. Through an arbitrary point P in space, we draw lines parallel to the coördinate axes, meeting the given planes in the points P_{yz}, P_{zx}, and P_{xy}. We can now give the following definition.

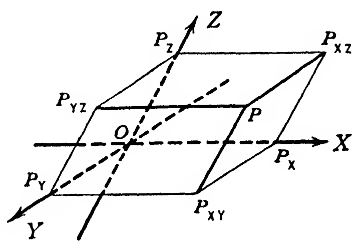

FIG. 14

DEFINITION III. **The (oblique) Cartesian coördinates of the point P are the lengths of the lines $P_{yz}P$, $P_{zx}P$, and $P_{xy}P$ measured in accordance with the units and directions specified for the X-, Y-, and Z-axes respectively.**

The coördinate planes, together with the planes determined by the lines $P_{yz}P$, $P_{zx}P$, and $P_{xy}P$, taken two at the time, form an oblique parallelopiped. This parallelopiped can be used conveniently to develop generalizations of some of the results obtained in Chapter III, just as these results themselves were found by the aid of the rectangular parallelopipeds, which we designated as the c.p. of a point and the c.p. of a pair of points (see Sections 30 and 32).

Notation. The coördinate frame which we have just described will be designated by the symbol O–XYZ. If the angles between

the coördinate axes have to be specified, we shall use the symbol O–XYZ–$\alpha\beta\gamma$, where $\alpha = \angle YOZ$, $\beta = \angle ZOX$, and $\gamma = \angle XOY$.

It will be supposed throughout that the units of measurement on the three axes of any Cartesian reference frame, and also those used on the axes of two such frames whose mutual relations are under consideration, are equal to each other.

61. Translation of Axes. We consider now the relations existing between the two sets of coördinates of a point P with reference to two Cartesian reference frames, whose axes are parallel; these may be rectangular or oblique frames. Let the two reference

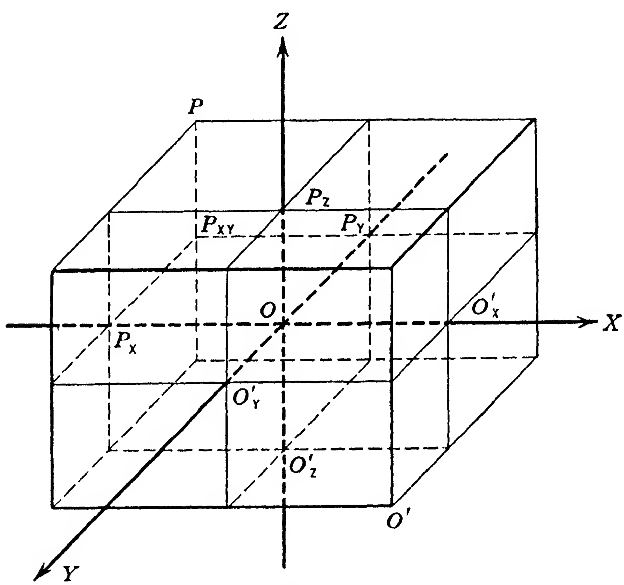

FIG. 15

frames be O–XYZ and O'–$X'Y'Z'$, and let the coördinates of O' with respect to O–XYZ be a, b, and c. For an arbitrary point $P(x, y, z)$ we construct now the c.p. of P and O' with respect to O–XYZ. Since the axes of the two frames are parallel, this parallelopiped will also be the c.p. of P with respect to O'–$X'Y'Z'$ and therefore its edges will be equal in unsigned length to the numerical values of the coördinates of P with respect to O'–$X'Y'Z'$, that is, of x', y', and z'. The X-, Y-, and Z-axes will meet the faces of this parallelopiped in the points P_x, O_x'; P_y, O_y'; and P_z, O_z' respec-

tively; and the segments $O_x'P_x$, $O_y'P_y$, and $O_z'P_z$ are equal to x', y', and z' respectively (see Fig. 15). We have now, independently of the positions of P and of the reference frame O'–$X'Y'Z'$, the following relations:

$$O_x'O + OP_x + P_xO_x' = 0, O_y'O + OP_y + P_yO_y' = 0, \quad \text{and}$$
$$O_z'O + OP_z + P_zO_z' = 0$$

and therefore

$$-a + x - x' = 0, \quad -b + y - y' = 0, \quad \text{and} \quad -c + z - z' = 0.$$

The result of the discussion can be summarized in the following theorem:

THEOREM 2. **The coördinates x, y, z of an arbitrary point P with reference to a Cartesian frame of reference O-XYZ, and the coördinates x', y', z' of the same point with reference to a parallel Cartesian frame of reference O'-$X'Y'Z'$, whose origin has the coördinates a, b, c with respect to O-XYZ satisfy the relations**

$$x' = x - a, \quad y' = y - b, \quad z' = z - c.$$

Remark 1. The coördinates x', y', and z' of the point O' are all 0; hence we find from the theorem just stated, that the coördinates of O' with reference to O–XYZ are a, b, c, as stated in the hypothesis of the theorem; this simple fact serves as a check on the formulas. Similarly, we find that the point O whose coördinates in the system O–XYZ are (0, 0, 0) has the coördinates $(-a, -b, -c)$ in the system O'–$X'Y'Z'$.

Remark 2. It should be noted that the formulas established in Theorem 2 are the same, independently of whether the two reference frames are oblique or rectangular.

62. Transformation from Oblique to Rectangular Axes. Before taking up the transformation of coördinates which results when we pass from one arbitrary frame of reference to another, we shall consider what happens when we change from an oblique frame of reference to a special associated rectangular frame. A system of rectangular axes can be superimposed upon a given oblique reference frame O–XYZ–$\alpha\beta\gamma$ by using in both systems the same XY-plane, the same X-axis, the same origin, and the same units of measurement. This is illustrated in Fig. 16, in which the axes of the rectangular frame are designated by OX', OY', and OZ'. To obtain the relations between the coördinates of an arbitrary point

P with respect to these two frames of reference, we make use of the projection method (see Section 36). The coördinate parallelopipeds of P with respect to these two sets of axes furnish a closed broken line, leading from O to P along the edges OP_x, P_xP_{xy}, $P_{xy}P$,

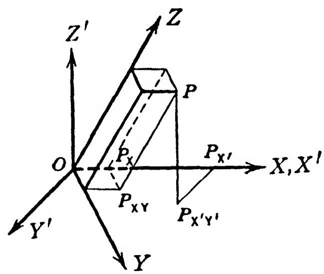

FIG. 16

and back from P to O along the edges PP_{xy}', $P_{xy}'P_x'$, $P_x'O$. We infer now from Theorem 12, Chapter III (see Section 36, page 62) that

$$\begin{aligned}\text{Proj}_{X'}OP_x + \text{Proj}_{X'}P_xP_{xy} + \text{Proj}_{X'}P_{xy}P + \text{Proj}_{X'}PP_{xy}'\\ + \text{Proj}_{X'}P_{xy}'P_x' + \text{Proj}_{X'}P_x'O = 0.\end{aligned}$$

To evaluate these projections, we make use of Theorem 11, Chapter III (see Section 36, page 62), remembering that the angles YOZ, ZOX, and XOY formed by the original axes are equal to α, β, and γ respectively, and that the X', Y', and Z' axes are mutually orthogonal. In this way we find that

$$x + y\cos\gamma + z\cos\beta - x' = 0.$$

By projecting the same path $OP_xP_{xy}PP_{xy}'P_x'O$ upon the Y'- and Z'- axes, we find

$$y\cos\angle YOY' + z\cos\angle ZOY' - y' = 0 \quad \text{and}$$
$$z\cos\angle ZOZ' - z' = 0.$$

Clearly $\angle YOY' = \dfrac{\pi}{2} - \angle XOY,$ so that $\cos\angle YOY' = \sin\gamma.$

To determine $\cos\angle ZOY'$ and $\cos\angle ZOZ'$, we make use of the result of Exercises 15 and 16, Section 38 (page 66), from which we find that

$$\cos\angle ZOY' = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\gamma} \quad \text{and}$$

$$\cos\angle ZOZ' = \frac{\pm 1}{\sin\gamma}\sqrt{1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma},$$

in which the $+$ or $-$ sign is to be used according as the Z-axis and the Z'-axis point toward the same side or toward opposite sides of the XY-plane. Substitution of these values in the preceding equations leads to the following theorem.

THEOREM 3. **If x, y, and z are the coördinates of an arbitrary point P in the Cartesian frame O-XYZ-$\alpha\beta\gamma$, and if x', y', and z' are the coördinates of the same point with reference to the orthogonal Cartesian frame O'-$X'Y'Z'$ in which the units of measurement, the origin, the X-axis, and the XY-plane are the same as the corresponding elements of the frame O-XYZ-$\alpha\beta\gamma$, then**

$$x' = x + y\cos\gamma + z\cos\beta, \quad y' = y\sin\gamma + \frac{z(\cos\alpha - \cos\beta\cos\gamma)}{\sin\gamma},$$

$$z' = \frac{\pm z(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma)^{\frac{1}{2}}}{\sin\gamma},$$

in which the plus or minus sign is to be used according as the two reference frames are of the same or of opposite type (see footnote on p. 50).

By means of the formulas of this theorem we can express the distance of a point from the origin of a system of oblique axes in terms of the oblique coördinates of this point. For

$$\begin{aligned}
OP^2 &= x'^2 + y'^2 + z'^2 \\
&= (x + y\cos\gamma + z\cos\beta)^2 + \\
&\quad \left[y\sin\gamma + \frac{z(\cos\alpha - \cos\beta\cos\gamma)}{\sin\gamma}\right]^2 \\
&\quad + \frac{z^2(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma)}{\sin^2\gamma} \\
&= x^2 + y^2(\cos^2\gamma + \sin^2\gamma) \\
&\quad + \frac{\begin{array}{c} z^2[\cos^2\beta\sin^2\gamma + (\cos\alpha - \cos\beta\cos\gamma)^2 + 1 - \cos^2\alpha - \\ \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma]\end{array}}{\sin^2\gamma} \\
&\quad + 2xy\cos\gamma + 2xz\cos\beta \\
&\quad + 2yz[\cos\gamma\cos\beta + \cos\alpha - \cos\beta\cos\gamma] \\
&= x^2 + y^2 + z^2 + 2yz\cos\alpha + 2zx\cos\beta + 2xy\cos\gamma.
\end{aligned}$$

COROLLARY 1. **The square of the distance from the origin of a reference frame O-XYZ-$\alpha\beta\gamma$ to a point $P(x, y, z)$ is equal to $x^2 + y^2 + z^2 + 2yz\cos\alpha + 2zx\cos\beta + 2xy\cos\gamma$.**

The expression for $\cos ZOZ'$ which was used in the proof of Theorem 3 enables us moreover to obtain a convenient formula

for the volume of the c.p. of the point $P(a, b, c)$ in an oblique reference frame. For the area of the base of this parallelopiped is equal to $ab\sin\gamma$ and its altitude is $c \cos ZOZ'$. This leads readily to the following result.

COROLLARY 2. **The volume of the coördinate parallelopiped of the point $P(a, b, c)$ in the reference frame O-XYZ-$\alpha\beta\gamma$ is equal to abc [1 − $\cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma]^{\frac{1}{2}} = abc\begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}^{\frac{1}{2}}$.**

63. Rotation of Axes. The projection method also enables us to determine in a direct manner the relations which connect the coördinates of one point in two Cartesian reference frames which have the same origin. We shall first develop these equations for the general case in which both systems are oblique and then obtain as a special case the formulas for two rectangular systems.

Let the systems be O–XYZ–$\alpha\beta\gamma$ and O–$X_1Y_1Z_1$–$\alpha_1\beta_1\gamma_1$, the units being the same in the two. Let the cosines of the angles formed by the axes of these systems be indicated in the following table:

	X	Y	Z
X_1	l_1	m_1	n_1
Y_1	l_2	m_2	n_2
Z_1	l_3	m_3	n_3

so that we have $\cos XOX_1 = l_1$, $\cos XOY_1 = l_2$, $\cos ZOY_1 = n_2$, etc.

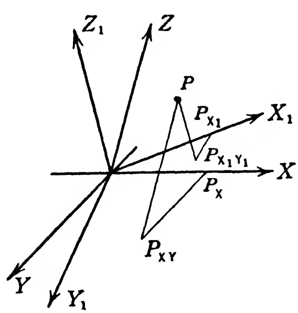

FIG. 17

We consider now the closed broken line which leads from O to P along the edges OP_x, P_xP_{xy} and $P_{xy}P$ of the c.p. of P in the system O–XYZ, and which returns from P to O along the edges $PP_{x_1y_1}$, $P_{x_1y_1}P_{x_1}$, $P_{x_1}O$ of the c.p. of P in the system O–$X_1Y_1Z_1$ (see Fig. 17). We project this closed broken line in turn on the axes OX, OY, and OZ, and then on the axes OX_1, OY_1, OZ_1. If we make use in each of these projections of Theorems 11 and 12 of Chapter III (see Section 36, page 62) and if we employ also the notation in-

troduced above for the angles between the two sets of axes, we find:

$$(1)\quad \begin{cases} x + y\cos\gamma + z\cos\beta - x_1 l_1 - y_1 l_2 - z_1 l_3 = 0, \\ x\cos\gamma + y + z\cos\alpha - x_1 m_1 - y_1 m_2 - z_1 m_3 = 0, \\ x\cos\beta + y\cos\alpha + z - x_1 n_1 - y_1 n_2 - z_1 n_3 = 0; \end{cases}$$

and

$$(2)\quad \begin{cases} x l_1 + y m_1 + z n_1 - x_1 - y_1\cos\gamma_1 - z_1\cos\beta_1 = 0, \\ x l_2 + y m_2 + z n_2 - x_1\cos\gamma_1 - y_1 - z_1\cos\alpha_1 = 0, \\ x l_3 + y m_3 + z n_3 - x_1\cos\beta_1 - y_1\cos\alpha_1 - z_1 = 0. \end{cases}$$

The system of equations (1) has a unique solution for x, y, z in terms of x_1, y_1, z_1; and the system (2) has a unique solution for x_1, y_1, z_1 in terms of x, y, z. These solutions may be written down by means of Cramer's rule (see Section 21, p. 37). For the coefficient determinants of these systems are equal respectively to

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix} \quad\text{and}\quad \begin{vmatrix} 1 & \cos\gamma_1 & \cos\beta_1 \\ \cos\gamma_1 & 1 & \cos\alpha_1 \\ \cos\beta_1 & \cos\alpha_1 & 1 \end{vmatrix};$$

and, by virtue of Corollary 2 of Theorem 3 (Section 62, page 118), the values of these determinants are equal to the squares of the volumes of the c.p.'s of the points (1, 1, 1) in the two systems. If these volumes are denoted by v and v_1 respectively, we find from the first system, that

$$x = \frac{1}{v} \times \begin{vmatrix} l_1x_1 + l_2y_1 + l_3z_1 & \cos\gamma & \cos\beta \\ m_1x_1 + m_2y_1 + m_3z_1 & 1 & \cos\alpha \\ n_1x_1 + n_2y_1 + n_3z_1 & \cos\alpha & 1 \end{vmatrix} = a_{11}x_1 + a_{12}y_1 + a_{13}z_1,$$

where

$$a_{11} = \frac{1}{v} \times \begin{vmatrix} l_1 & \cos\gamma & \cos\beta \\ m_1 & 1 & \cos\alpha \\ n_1 & \cos\alpha & 1 \end{vmatrix}, \quad a_{12} = \frac{1}{v} \times \begin{vmatrix} l_2 & \cos\gamma & \cos\beta \\ m_2 & 1 & \cos\alpha \\ n_2 & \cos\alpha & 1 \end{vmatrix},$$

$$a_{13} = \frac{1}{v} \times \begin{vmatrix} l_3 & \cos\gamma & \cos\beta \\ m_3 & 1 & \cos\alpha \\ n_3 & \cos\alpha & 1 \end{vmatrix},$$

(see Theorem 8, Chapter I, Section 5, page 9). Similar results are obtained for y and z, the trinomial elements now appearing in the second and third columns. And from the system (2) we obtain

the solution

$$y_1 = \frac{1}{v_1} \times \begin{vmatrix} 1 & l_1x+m_1y+n_1z & \cos\beta_1 \\ \cos\gamma_1 & l_2x+m_2y+n_2z & \cos\alpha_1 \\ \cos\beta_1 & l_3x+m_3y+n_3z & 1 \end{vmatrix} = b_{21}x+b_{22}y+b_{23}z,$$

where now

$$b_{21} = \frac{1}{v_1} \times \begin{vmatrix} 1 & l_1 & \cos\beta_1 \\ \cos\gamma_1 & l_2 & \cos\alpha_1 \\ \cos\beta_1 & l_3 & 1 \end{vmatrix}, \quad b_{22} = \frac{1}{v_1} \times \begin{vmatrix} 1 & m_1 & \cos\beta_1 \\ \cos\gamma_1 & m_2 & \cos\alpha_1 \\ \cos\beta_1 & m_3 & 1 \end{vmatrix},$$

$$b_{23} = \frac{1}{v_1} \times \begin{vmatrix} 1 & n_1 & \cos\beta_1 \\ \cos\gamma_1 & n_2 & \cos\alpha_1 \\ \cos\beta_1 & n_3 & 1 \end{vmatrix},$$ with similar results for

x_1 and z_1. The reader should have little difficulty in writing out these further results. We state the following conclusion of our discussion.

THEOREM 4. **If x, y, z are the coördinates of an arbitrary point P with respect to a reference frame O-XYZ-$\alpha\beta\gamma$, and x_1, y_1, z_1 are the coördinates of the same point in the reference frame O-$X_1Y_1Z_1$-$\alpha_1\beta_1\gamma_1$, of which the X_1-, Y_1-, and Z_1-axes make with the axes of O-XYZ angles whose cosines are l_1, m_1, n_1; l_2, m_2, n_2 and l_3, m_3, n_3 respectively, then**

$$\begin{aligned} x &= a_{11}x_1 + a_{12}y_1 + a_{13}z_1, \\ y &= a_{21}x_1 + a_{22}y_1 + a_{23}z_1, \\ z &= a_{31}x_1 + a_{32}y_1 + a_{33}z_1; \end{aligned} \quad \textbf{and} \quad \begin{aligned} x_1 &= b_{11}x + b_{12}y + b_{13}z, \\ y_1 &= b_{21}x + b_{22}y + b_{23}z, \\ z_1 &= b_{31}x + b_{32}y + b_{33}z. \end{aligned}$$

Here $a_{ij}(i, j = 1, 2, 3)$ is the value of the determinant obtained from the determinant $v = \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}$ by replacing the ith column by l_j, m_j, n_j, divided by the value of v; and the coefficient $b_{ij}(i, j = 1, 2, 3)$ is the quotient by the value of the determinant $v_1 = \begin{vmatrix} 1 & \cos\gamma_1 & \cos\beta_1 \\ \cos\gamma_1 & 1 & \cos\alpha_1 \\ \cos\beta_1 & \cos\alpha_1 & 1 \end{vmatrix}$ of the determinant obtained from v_1 by replacing the ith column by the jth column of the determinant $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$.

Remark. The reader should write out in full the values of the coefficients a_{ij} and b_{ij} in the form of the determinants, which in

order to save space have merely been described in the statement of the theorem.

The formulas established in this theorem take a particularly simple form in case both reference frames are rectangular. For in that case, $\alpha = \beta = \gamma = \alpha_1 = \beta_1 = \gamma_1 = \frac{\pi}{2}$, hence $\cos\alpha = \cos\beta = \cos\gamma = \cos\alpha_1 = \cos\beta_1 = \cos\gamma_1 = 0$; moreover the numbers l_1, m_1, n_1; l_2, m_2, n_2 and l_3, m_3, n_3 become the direction cosines of the axes OX_1, OY_1, and OZ_1 with respect to $O\text{-}XYZ$ respectively. The reader should have no difficulty in obtaining the formulas for this special case which are stated in the following theorem; these formulas, more than those of Theorem 4, are the ones which we shall have frequent occasion to use in our further work.

THEOREM 5. **If x, y, z are the coördinates of a point P with respect to a rectangular Cartesian frame of reference $O\text{-}XYZ$, and x_1, y_1, z_1 the coördinates of the same point with respect to another rectangular frame $O\text{-}X_1Y_1Z_1$, of which the X_1-, Y_1-, and Z_1-axes have in $O\text{-}XYZ$ direction cosines λ_1, μ_1, ν_1; λ_2, μ_2, ν_2; and λ_3, μ_3, ν_3 respectively, then**

$$\begin{aligned} x &= \lambda_1 x_1 + \lambda_2 y_1 + \lambda_3 z_1, \\ y &= \mu_1 x_1 + \mu_2 y_1 + \mu_3 z_1, \\ z &= \nu_1 x_1 + \nu_2 y_1 + \nu_3 z_1; \end{aligned} \quad \textbf{and} \quad \begin{aligned} x_1 &= \lambda_1 x + \mu_1 y + \nu_1 z, \\ y_1 &= \lambda_2 x + \mu_2 y + \nu_2 z, \\ z_1 &= \lambda_3 x + \mu_3 y + \nu_3 z. \end{aligned}$$

64. Exercises.

1. Set up the equations for the transformation of coördinates resulting from translating the axes to a new origin whose coördinates in a system $O\text{-}XYZ$ are $-3, 5, 2$.

2. Determine the equation of the sphere $x^2 + y^2 + z^2 = 9$ with respect to a new frame of reference obtained by translating the original axes to the new origin $O'(-2, -1, 3)$.

3. Show that the planes determined by the equations $3x - 6y + 2z = 0$, $2x + y = 0$ and $2x - 4y - 15z = 0$ are mutually perpendicular, and that they pass through the origin. Establish the formulas for the transformation of coördinates which results when these planes are taken respectively as the Y_1Z_1-, the Z_1X_1-, and the X_1Y_1-plane of a new frame of reference.

4. Solve the same problem for the planes $2x - y + 2z = 0$, $x - 2y - 2z = 0$, $2x + 2y - z = 0$.

5. Apply the formulas obtained in Exercise 4 to determine the equations in the new reference frame of the loci of the following equations:

(a) $y^2 + z^2 = 3$; (b) $x^2 + y^2 + z^2 = 4$; (c) $2x^2 - 5y^2 - 4z^2 = 10$; (d) $ax + by + cz + d = 0$

6. Express the distance between two points in terms of their coördinates in a system of oblique axes.

7. Determine the volume of the coördinate parallelopiped of two points in an oblique frame of reference.

8. Prove that, if θ_1, θ_2, and θ_3 are the angles which a line l makes with the axes of a reference frame $O\text{–}XYZ\text{–}\alpha\beta\gamma$, then

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta & \cos\theta_1 \\ \cos\gamma & 1 & \cos\alpha & \cos\theta_2 \\ \cos\beta & \cos\alpha & 1 & \cos\theta_3 \\ \cos\theta_1 & \cos\theta_2 & \cos\theta_3 & 1 \end{vmatrix} = 0.$$

Hint: If l coincides with one of the coördinate axes, the formula can readily be verified. If l does not coincide with any of the axes, take a point P on l, so that $OP = 1$ and project the closed broken line $OP_xP_{xy}PO$ on the axes and on l; from the resulting equations the desired formula should follow.

9. Show that, in case $\alpha = \beta = \gamma = \frac{\pi}{2}$, the formula of the preceding exercise reduces to that given in Theorem 7, Chapter III (Section 33, page 56).

10. Show that if the formula of Exercise 8 reduces, for *every* line l, to the formula of Theorem 7, Chapter III, then $\alpha = \beta = \gamma = \frac{\pi}{2}$.

65. Rotation of Axes, *continued.* The formulas obtained in Theorems 4 and 5 appear to contain a large number of parameters; but these are not all independent parameters. For the a_{ij} and b_{ij} of Theorem 4, and the λ_i, μ_i and ν_i of Theorem 5 can not be chosen arbitrarily if the formulas are to represent a rotation of axes. This can be seen most readily if we observe that the expressions for the distance from O to an arbitrary point P should be the same in any two frames of reference which have the same origin. Hence it follows that if the parameters in the formulas of Theorem 4 are properly selected, then we must have, in view of Corollary 1 of Theorem 3 (Section 62, page 117):

$$\begin{aligned} &x^2 + y^2 + z^2 + 2\,yz\cos\alpha + 2\,zx\cos\beta + 2\,xy\cos\gamma \\ &\quad = x_1^2 + y_1^2 + z_1^2 + 2\,y_1z_1\cos\gamma_1 + 2\,z_1x_1\cos\beta_1 + 2\,x_1y_1\cos\gamma_1 \end{aligned}$$

for all values of x, y, and z, if for x_1, y_1, and z_1 we substitute the expressions given in Theorem 4. If, in particular, both reference frames are rectangular, we find by using Theorem 5 that

$$\begin{aligned} &(\lambda_1x + \mu_1y + \nu_1z)^2 + (\lambda_2x + \mu_2y + \nu_2z)^2 + (\lambda_3x + \mu_3y + \nu_3z)^2 \\ &\quad = x^2 + y^2 + z^2 \end{aligned}$$

for all values of x, y, and z.

If we carry out the squaring of the trinomials on the left-hand side, and equate the coefficients of like terms on the two sides, we

are led to the conclusion that, if the formulas of Theorem 5 do indeed represent a rotation of a rectangular reference frame, then the parameters λ_i, μ_i, ν_i, $i = 1, 2, 3$, must satisfy the following relations:

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1, \quad \mu_1^2 + \mu_2^2 + \mu_3^2 = 1, \quad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1$$

and
$$\mu_1\nu_1 + \mu_2\nu_2 + \mu_3\nu_3 = 0, \quad \nu_1\lambda_1 + \nu_2\lambda_2 + \nu_3\lambda_3 = 0,$$
$$\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = 0.$$

Conversely, in case these conditions hold, the equations of Theorem 5 will carry a given rectangular frame over into another rectangular frame with the same origin. For, by virtue of the first three of the above relations, we can then take $\lambda_1, \lambda_2, \lambda_3$; μ_1, μ_2, μ_3, and ν_1, ν_2, ν_3 as the direction cosines of three lines through the origin; and it follows from the last three relations that these lines are mutually perpendicular. The equations of Theorem 5 represent then the transformation to the new rectangular reference frame of which these three lines are the axes. Hence we have established the following theorem.

THEOREM 6. **If x_1, y_1, z_1 represent the coördinates of a point with respect to a rectangular reference frame, the necessary and sufficient conditions that the equations**

$$x_1 = \lambda_1 x + \mu_1 y + \nu_1 z, \quad y_1 = \lambda_2 x + \mu_2 y + \nu_2 z, \quad z_1 = \lambda_3 x + \mu_3 y + \nu_3 z$$

shall represent a transformation to another rectangular reference frame, are that

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1, \quad \mu_1^2 + \mu_2^2 + \mu_3^2 = 1, \quad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1$$

and that

$$\mu_1\nu_1 + \mu_2\nu_2 + \mu_3\nu_3 = 0, \quad \nu_1\lambda_1 + \nu_2\lambda_2 + \nu_3\lambda_3 = 0. \quad \lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = 0.$$

Remark 1. A transformation which satisfies the conditions of Theorem 6 is called, with obvious justification, an **orthogonal transformation.**

Remark 2. With the aid of Theorem 6, it becomes easy to verify that the two sets of equations in Theorem 5 are equivalent. For if we multiply those of the second set by $\lambda_1, \lambda_2, \lambda_3$ respectively and then substitute the results in the first equation of the first set, we find:

$$\begin{aligned} x &= \lambda_1(\lambda_1 x + \mu_1 y + \nu_1 z) + \lambda_2(\lambda_2 x + \mu_2 y + \nu_2 z) + \lambda_3(\lambda_3 x + \mu_3 y + \nu_3 z) \\ &= (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)x + (\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3)y + (\lambda_1\nu_1 + \lambda_2\nu_2 + \lambda_3\nu_3)z \\ &= x; \end{aligned}$$

and the other equations of the first set are verified in similar manner.

Remark 3. It should be clear that the conditions of Theorem 6 can also be put in the equivalent form:

$$\lambda_1^2 + \mu_1^2 + \nu_1^2 = \lambda_2^2 + \mu_2^2 + \nu_2^2 = \lambda_3^2 + \mu_3^2 + \nu_3^2 = 1$$

and

$$\lambda_2\lambda_3 + \mu_2\mu_3 + \nu_2\nu_3 = \lambda_3\lambda_1 + \mu_3\mu_1 + \nu_3\nu_1 = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0.$$

The equivalence of the two sets of equations in Theorem 5, observed in Remark 2 above, leads to another interesting result. Since neither set of axes consists of coplanar lines, their orientation determinant (the determinant formed from their nine direction cosines, see Exercise 12, Section 56, page 106) does not vanish. Hence the equations of the first set can be solved for x_1, y_1, and z_1 by Cramer's rule. If we denote the value of the orientation determinant by D, we find, for example,

$$x_1 = \frac{1}{D} \times \begin{vmatrix} x & \lambda_2 & \lambda_3 \\ y & \mu_2 & \mu_3 \\ z & \nu_2 & \nu_3 \end{vmatrix} = \frac{1}{D}\left\{\begin{vmatrix} \mu_2 & \mu_3 \\ \nu_2 & \nu_3 \end{vmatrix} x + \begin{vmatrix} \nu_2 & \nu_3 \\ \lambda_2 & \lambda_3 \end{vmatrix} y + \begin{vmatrix} \lambda_2 & \lambda_3 \\ \mu_2 & \mu_3 \end{vmatrix} z\right\}.$$

But this value of x_1 must be identical with the value furnished by the first equation of the second set, for every value of x, y, and z. Consequently, the coefficients of x, y, and z in the two expressions for x_1 must be equal, each to each. Hence we have

$$\lambda_1 D = \begin{vmatrix} \mu_2 & \mu_3 \\ \nu_2 & \nu_3 \end{vmatrix}, \quad \mu_1 D = \begin{vmatrix} \nu_2 & \nu_3 \\ \lambda_2 & \lambda_3 \end{vmatrix}, \quad \nu_1 D = \begin{vmatrix} \lambda_2 & \lambda_3 \\ \mu_2 & \mu_3 \end{vmatrix}.$$

If these equations are squared and added, we find:

$$D^2 = \begin{vmatrix} \mu_2 & \mu_3 \\ \nu_2 & \nu_3 \end{vmatrix}^2 + \begin{vmatrix} \nu_2 & \nu_3 \\ \lambda_2 & \lambda_3 \end{vmatrix}^2 + \begin{vmatrix} \lambda_2 & \lambda_3 \\ \mu_2 & \mu_3 \end{vmatrix}^2.$$

But the right-hand side of this equation represents the square of the sine of the angle between the lines whose direction cosines are λ_2, μ_2, ν_2 and λ_3, μ_3, ν_3, that is, between OY_1 and OZ_1; it is therefore equal to 1 (see Theorem 14, Chapter III, Section 36, page 64). We have therefore obtained the following result.

THEOREM 7. **The value of the orientation determinant** $\begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}$ **of three mutually perpendicular directed lines is equal to $+1$ or to -1.**

Remark. It follows from Theorem 7 that if two of the three directed lines are interchanged, the value of the orientation determinant changes sign; this will also happen if the direction on one of the lines, or on all three, is changed. This suggests that whether D is $+1$ or -1 depends upon whether or not the three lines whose direction cosines are given by the elements in its rows, taken in the order of these rows, form a reference frame of the same type as the frame with respect to which their direction cosines are taken (see footnote on page 50). This is indeed the case, but a satisfactory proof of this fact can not be given without a more extended discussion than can find a place in this book; for it involves considerations of continuity. We shall therefore not pursue this question.

We shall likewise omit a discussion of the conditions which the parameters a_{ij} and b_{ij} in the equations of Theorem 4 must satisfy in order that these equations may represent a transformation from one reference frame to another with the same origin.

66. Linear Transformation. Plane Sections of a Surface. If we combine the results of Theorems 2 and 5 (see Sections 61, page 115, and 63, page 121), we obtain formulas for the transformation of coördinates which occurs when we pass from one rectangular Cartesian reference frame to another, keeping the units unchanged. For such a change can always be accomplished by a translation and a rotation. Suppose that, with reference to the frame $O\text{-}XYZ$, the coördinates of the new origin O_1 are a, b, c; and that the direction cosines of the axes O_1X_1, O_1Y_1, and O_1Z_1 are $\lambda_1, \mu_1, \nu_1, \lambda_2, \mu_2, \nu_2$, and λ_3, μ_3, ν_3 respectively. Starting with $O\text{-}XYZ$, we translate the axes to the new origin O_1; this leads to the reference frame $O_1\text{-}X_2Y_2Z_2$ (see Fig. 18). From this we make the transition to the frame $O_1\text{-}X_1Y_1Z_1$ by a rotation of axes. It follows, from Theorem 2, that $x = x_2 + a$, $y = y_2 + b$, $z = z_2 + c$; and from Theorem 5,

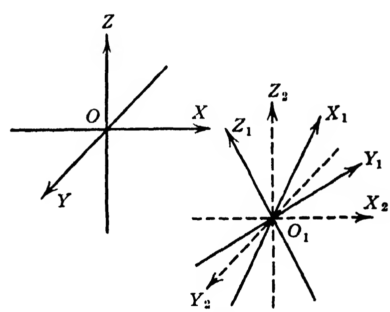

FIG. 18

that $x_2 = \lambda_1 x_1 + \lambda_2 y_1 + \lambda_3 z_1$, $y_2 = \mu_1 x_1 + \mu_2 y_1 + \mu_3 z_1$, $z_2 = \nu_1 x_1 + \nu_2 y_1 + \nu_3 z_1$. We obtain therefore the following theorem.

THEOREM 8. **If x, y, z and x_1, y_1, z_1 are the coördinates of an arbitrary point P with reference to the rectangular Cartesian frames O-XYZ and O_1-$X_1Y_1Z_1$ respectively, and when, with respect to O-XYZ, the coördinates of O_1 are a, b, c, and the direction cosines of OX_1, OY_1, and OZ_1 are $\lambda_1, \mu_1, \nu_1, \lambda_2, \mu_2, \nu_2$, and λ_3, μ_3, ν_3 respectively, then**

$$\begin{aligned} x &= \lambda_1 x_1 + \lambda_2 y_1 + \lambda_3 z_1 + a, \\ y &= \mu_1 x_1 + \mu_2 y_1 + \mu_3 z_1 + b, \\ z &= \nu_1 x_1 + \nu_2 y_1 + \nu_3 z_1 + c. \end{aligned}$$

Remark 1. If we solve these equations for x_1, y_1, and z_1, and make use of the relations established in the proof of Theorem 6 (see Section 65, page 123), we find that

$$\begin{aligned} x_1 &= \lambda_1(x - a) + \mu_1(y - b) + \nu_1(z - c), \\ y_1 &= \lambda_2(x - a) + \mu_2(y - b) + \nu_2(z - c), \\ z_1 &= \lambda_3(x - a) + \mu_3(y - b) + \nu_3(z - c). \end{aligned}$$

It will be worth while for the reader to deduce this result by direct application of Theorems 2 and 5.

Remark 2. A transformation of the frame of reference such as we have discussed in the preceding paragraphs will be called a **rigid transformation.** The algebraic transformation of coördinates which corresponds to it is called a transformation of the first degree, or a **linear transformation.**

COROLLARY 1. **The degree of a polynomial in x, y, z, such as $f(x, y, z)$, is the same as that of the polynomial $f_1(x_1, y_1, z_1)$ obtained from $f(x, y, z)$ by a linear transformation.**

Proof. Since the expressions to be substituted for x, y, and z are of the first degree in x_1, y_1, and z_1, it should be clear that the degree of f_1 can not exceed that of f. But since f can be obtained from f_1 by substituting for x_1, y_1, and z_1 the linear functions of x, y, and z stated in Remark 1, the degree of f can not exceed that of f_1. Therefore the degrees of the two polynomials are equal.

Remark. The transformation of coördinates which corresponds to a rotation of axes carries a homogeneous polynomial in x, y, z over into a homogeneous polynomial of the same degree in x_1, y_1, z_1.

We are now prepared to take up a question of interest and importance, namely, to determine the character of the curve of

intersection of a surface with an arbitrary plane. If one of the variables, let us say by way of example y, is eliminated between the equation of the surface $f(x, y, z) = 0$ and that of the plane $ax + by + cz + d = 0$, we obtain an equation, say $F(x, z) = 0$, whose space locus is the cylindrical surface parallel to the Y-axis, which projects the curve of intersection of surface and plane upon the ZX-plane; and whose plane locus is the projection of this curve upon the ZX-plane (compare Theorems 1 and 2, Chapter IV, Section 40, pages 69, 71).

When the intersecting plane is parallel to one of the coördinate planes the curve of intersection is congruent to its projection upon that coördinate plane; in that case our question can be answered immediately by the methods of Plane Analytical Geometry. But when the intersecting plane is in a general position, these two curves will not be congruent. The question can then be answered, as is suggested clearly by the answer in the special case, by first making a transformation of coördinates to a new reference frame, of which one of the coördinate planes is parallel to the given plane. How is such a transformation determined?

Let us propose so to transform a given frame of reference O–XYZ to a new frame O_1–$X_1Y_1Z_1$ that a plane whose equation in normal form (see Section 44) is $\lambda x + \mu y + \nu z - p = 0$ shall be parallel to the X_1Y_1-plane. The necessary and sufficient condition for this is that the direction cosines of the Z_1-axis shall be λ, μ, ν (see Theorem 7, Chapter IV, Section 44, page 78). It follows therefore from Theorem 8 that the desired transformation will be accomplished if we put

$$x = \lambda_1 x_1 + \lambda_2 y_1 + \lambda z_1 + a, \quad y = \mu_1 x_1 + \mu_2 y_1 + \mu z_1 + b,$$
$$z = \nu_1 x_1 + \nu_2 y_1 + \nu z_1 + c,$$

where λ_1, μ_1, ν_1, λ_2, μ_2, ν_2, and a, b, c are arbitrary, save for the restrictions imposed by Theorem 6. This arbitrariness in the choice of some of the coefficients in the equations of transformation corresponds to the fact that the position of the origin and that of the axes O_1X_1, O_1Y_1 have not yet been specified. When these specifications have been made the equations of transformation can be completely determined (see the Examples below).

In view of Corollary 1 (see page 126) the equation of the given surface in the new reference frame will have the same degree as

the original equation of the surface. Since the equation of the plane section of the surface is obtained from the new equation of the surface by replacing one of the variables by a constant, the degree of the equation of the curve of intersection will not exceed the degree of the equation of the surface. For convenience of reference, we record this fact as follows.

COROLLARY 2. **The degree of the plane equation of the section of a surface made by an arbitrary plane does not exceed the degree of the equation of the surface.**

Examples.

1. To determine the curve of intersection of the sphere $x^2 + y^2 + z^2 = 9$ with the plane $3x - 4y + 12z - 2 = 0$, we reduce the equation of the plane to the normal form $\frac{3x}{13} - \frac{4y}{13} + \frac{12z}{13} - \frac{2}{13} = 0$. Hence we have $\lambda = \frac{3}{13}$, $\mu = -\frac{4}{13}$, $\nu = \frac{12}{13}$. From the conditions of Theorem 6 (see Section 65, page 123), it follows that λ_1, μ_1, ν_1, and λ_2, μ_2, ν_2 must be so chosen that

$$(1)\quad 3\lambda_1 - 4\mu_1 + 12\nu_1 = 0 \quad \text{and} \quad \lambda_1^2 + \mu_1^2 + \nu_1^2 = 1$$

and that

$$(2)\quad 3\lambda_2 - 4\mu_2 + 12\nu_2 = 0,\ \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0,\ \text{and}\ \lambda_2^2 + \mu_2^2 + \nu_2^2 = 1.$$

If one of the coefficients λ_1, μ_1, or ν_1 is chosen arbitrarily and the others are determined so as to satisfy the equations (1), then the remaining coefficients λ_2, μ_2, ν_2 are completely fixed by equations (2), provided that λ_1, μ_1, and ν_1 are so selected that the rank of the coefficient matrix of the first two of equations (2) is 2. If we take $\nu_1 = 0$, we find $\lambda_1 : \mu_1 = 4 : 3$ and hence $\lambda_1 = \frac{4}{5}$, $\mu_1 = \frac{3}{5}$, $\nu_1 = 0$. For the determination of λ_2, μ_2, ν_2 we have then the conditions $3\lambda_2 - 4\mu_2 + 12\nu_2 = 0$ and $4\lambda_2 + 3\mu_2 = 0$; we find therefore, by using Theorem 4, Chapter II (see Section 25, page 41), that $\lambda_2 : \mu_2 \cdot \nu_2 = -36 : 48 : 25$ and hence that $\lambda_2 = -\frac{36}{65}$, $\mu_2 = \frac{48}{65}$, $\nu_2 = \frac{25}{65}$. If, finally, we take $a = b = c = 0$, we obtain the following equations of transformation:

$$x = \frac{4x_1}{5} - \frac{36y_1}{65} + \frac{3z_1}{13},\quad y = \frac{3x_1}{5} + \frac{48y_1}{65} - \frac{4z_1}{13},\quad z = \frac{5y_1}{13} + \frac{12z_1}{13}.$$

It follows now from the discussion at the beginning of Section 65 (see page 122) that these equations of transformation must carry the equation of the sphere over into the equation $x_1^2 + y_1^2 + z_1^2 = 9$. The equation of the given plane $3x - 4y + 12z - 2 = 0$ becomes:

$$3\left(\frac{4x_1}{5} - \frac{36y_1}{65} + \frac{3z_1}{13}\right) - 4\left(\frac{3x_1}{5} + \frac{48y_1}{65} - \frac{4z_1}{13}\right) + 12\left(\frac{5y_1}{13} + \frac{12z_1}{13}\right) - 2 = 0$$

and this reduces to the simple equation $13z_1 = 2$.

Hence the curve of intersection of sphere and plane is congruent to the curve in the X_1Y_1-plane whose equation is obtained by eliminating z_1 from the equations $x_1^2 + y_1^2 + z_1^2 = 9$ and $13\,z_1 = 2$, that is, congruent to the plane locus of the equation $169\,x_1^2 + 169\,y_1^2 = 1517$. The curve is therefore a circle. That the result would be a circle was evident from the fact that every plane cuts a sphere in a circle; the derivation which we have given serves to illustrate the method which can be used also in cases in which the result could not be so easily predicted.

2. The curve of intersection of the plane $3\,x - 4\,y + 12\,z - 2 = 0$, used in Example 1, with the surface $x^2 + y^2 + 2\,z^2 = 9$ is obtained by using the same equations of transformation that were used above. The equation of the surface now becomes

$$x_1^2 + y_1^2 + z_1^2 + \left(\frac{5\,y_1}{13} + \frac{12\,z_1}{13}\right)^2 = 9.$$

If this equation is solved simultaneously with the equation of the plane, $13\,z_1 = 2$, we find that the curve of intersection of the surface and the plane is congruent with the plane locus of the equation

$$169\,x_1^2 + 169\,y_1^2 + 4 + 169\left(\frac{5\,y_1}{13} + \frac{24}{169}\right)^2 = 1521$$

This equation reduces to $169\,x_1^2 + 194\,y_1^2 + \dfrac{240\,y_1}{13} - 1513\tfrac{100}{169} = 0$; the curve is therefore an ellipse.

67. Exercises.

1. Set up the equations for a transformation of coördinates which carries the plane $2\,x - y + 2\,z - 5 = 0$ over into the Z_1X_1-plane.

2. Determine the equations of transformation when the plane $x + 2\,y - 2\,z + 4 = 0$ is to become the Y_1Z_1-plane; the line $x + 2\,y - 2\,z + 4 = 0$, $3\,x - y + z - 3 = 0$ becomes the Z_1-axis; and the point $(\frac{2}{7}, -\frac{1}{7}, 2)$ is the new origin.

3. Set up the equations for a transformation of coördinates which will carry the line $3\,x + 4\,y - 2\,z = 0$, $z = 0$ into the X_1-axis.

4. Determine the volume of the coördinate parallelopiped of the point P whose coördinates are $x = -4$, $y = 2$, $z = -1$ in the reference frame in which the planes $2\,x - 2\,y + z = 0$, $y + z = 0$ and $x - z = 0$ are the Y_1Z_1-, Z_1X_1-, and X_1Y_1-planes respectively.

5. Determine a plane equation of the curve of intersection of the plane $6\,x - 2\,y + 3\,z - 4 = 0$ with each of the following surfaces:

(*a*) $3\,x^2 + 2\,y^2 - 3\,z^2 + 18 = 0$; (*b*) $4\,x^2 - y^2 = 6\,z$;
(*c*) $3\,xy + 4\,yz - 2\,zx = 10$.

6. Determine, for each of the axes of the new frame of reference introduced in Exercise 3, Section 64 (page 121), such a direction that the new frame is of the same type as the original frame; also such directions that the new frame is of the opposite type.

7. Solve the corresponding problem for the reference frame introduced in Exercise 4, Section 64.

8. Set up the equations of transformation for the transition from a reference frame $O\text{-}XYZ$ to a new frame whose origin has in $O\text{-}XYZ$ the coördinates $5, 3, -2$ and whose X-, Y-, and Z-axes have in $O\text{-}XYZ$ direction cosines which are proportional to $4, -8, 1$; $3, 2, 4$, and to $34, 13$ and -32 respectively. Decide how to select the directions on the new axes to make the new frame of the same type as $O\text{-}XYZ$.

9. Interpret geometrically the transformation of coördinates which is determined by the equations $3x = 2x_1 - y_1 - 2z_1$, $3y = x_1 - 2y_1 + 2z_1 - 6$, $3z = 2x_1 + 2y_1 + z_1 + 3$.

10. Prove that the curve in which an arbitrary plane meets the locus of an equation of the second degree in x, y, and z is a conic section or a straight line.

CHAPTER VI

GENERAL PROPERTIES OF SURFACES AND CURVES

Before undertaking a somewhat detailed study of the loci of equations of the second degree in x, y, and z, we shall consider in the present chapter a few general properties of surfaces and curves. Our attention will be restricted almost entirely to the loci of equations of the form $f(x, y, z) = 0$, whose left-hand side is a polynomial in the three variables. The case in which not all three of the variables are actually present in the equation has been considered in an earlier chapter (Section 40). We recall also Definitions I and II of Chapter IV (Section 39).

68. Surfaces of Revolution.

DEFINITION I. **A *surface of revolution* is a surface that can be generated by revolving a plane curve about a line in its plane. The line *a* around which the revolution takes place is called the *axis of revolution* of the surface; the plane curve, in any of its positions, is called a *meridian curve*.**

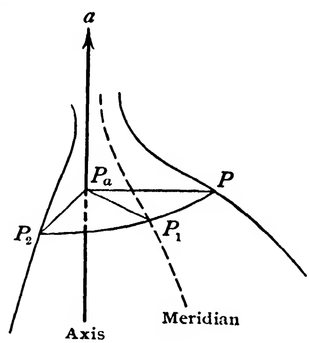

FIG. 19

It should be clear that all meridian curves are congruent plane curves and that the planes in which they lie constitute a pencil of planes through the axis of revolution.*

Every point P on the given curve, see Fig. 19, describes a circle, whose center is the projection P_a of P on the axis of revolution and whose radius is P_aP; these circles, which lie in planes perpendicular to the axis of revolution, are called **parallel circles.**

* The reader will notice that the simplest curved surfaces studied in elementary Solid Geometry, spheres, cones, cylinders, as well as the surfaces of a large number of manufactured articles in daily use, such as teacups, lampshades, hats, are, exactly or approximately, surfaces of revolution. Is there a possible simple reason for this?

Our principal problem is to determine the equation of a surface of revolution when we are given the equations of a meridian curve and those of the axis of revolution. In case the axis of revolution does not coincide with one of the coördinate axes, we can always transform the frame of reference in such a manner as to make one of the new coördinate axes coincide with the axis of revolution; and there is evidently no loss in generality if we assume that the meridian curve from which we start lies in one of the coördinate planes through the axis of revolution. We shall therefore suppose that the given curve lies in the XZ-plane and we shall seek the equation of the surface obtained by revolving this curve about the X-axis (Fig. 20). Let the equations of the meridian curve be $f(x, z) = 0$, $y = 0$; and let $P(x, y, z)$ be an arbitrary point on the surface. The parallel circle through P will cut the initial meridian in a point $Q(x', 0, z')$. It should now be easy to see that $x' = x$, and $z' = P_xP = \sqrt{z^2 + y^2}$. Since Q is a point on the meridian, its coördinates satisfy the equation $f(x, z) = 0$. Consequently, the coördinates of P satisfy the equation $f(x, \sqrt{y^2 + z^2}) = 0$. Conversely, if the coördinates of a point P' satisfy this equation, then the coördinates of the point Q' in which the circle through P' with center at P_x' and radius $P_x'P'$ cuts the XZ-plane, will satisfy the equations $f(x, z) = 0$ and $y = 0$; the point Q' will therefore lie on the given curve and the point P' on the surface of revolution which this curve generates when revolving about the X-axis. We have therefore reached the following conclusion:

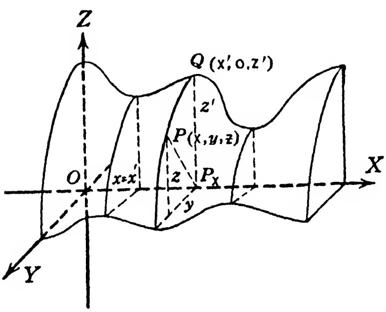

FIG. 20

THEOREM 1. **The equation of the surface of revolution generated when a plane curve in the ZX-plane revolves about the X-axis, is obtained by replacing z in the plane equation of this curve by $\sqrt{y^2 + z^2}$ and then rationalizing the equation.**

It should be a simple matter to state similar conclusions for the surface of revolution that is obtained when a curve in any co-

ördinate plane revolves about either axis in that plane (see Section 69).

Examples.

1. The equation of the surface generated when a parabola in the XY-plane, whose equations are $y^2 = 4\,ax$ and $z = 0$, is revolved about the X-axis, is obtained by replacing y in the plane equation of the curve by $\sqrt{y^2 + z^2}$ and then rationalizing. The surface is called a **paraboloid of revolution**; its equation is $y^2 + z^2 = 4\,ax$ (see Fig. 21).

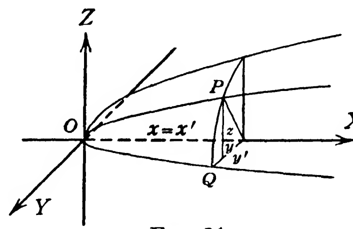

FIG. 21

2. Of especial interest to us are the surfaces of revolution generated when the conic sections are revolved about one of the axes of the curve. The shapes of these surfaces can easily be pictured. By taking the equations of the curves in the standard forms, familiar from Plane Analytical Geometry, we obtain readily the following results:

Meridian curve	Axis of revolution	Equation of surface of revolution	Name
(1) $x^2 + y^2 = a^2$, Circle	X-, or Y-axis	$x^2 + y^2 + z^2 = a^2$	Sphere.
(2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, Ellipse $a > b$	X-axis	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$	Ellipsoid of revolution; prolate spheroid.
(3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, Ellipse $a > b$	Y-axis	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$	Ellipsoid of revolution; oblate spheroid.
(4) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, Hyperbola	X-axis	$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{b^2} = 1$	Hyperboloid of revolution of two sheets.

The surface that is obtained in this case consists of two parts entirely separate from each other; they are called the two sheets (nappes) of the surface — compare also the Remark on page 136.

(5) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, Hyperbola	Y-axis	$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$	Hyperboloid of revolution of one sheet.
(6) $y^2 = 4\,ax$, Parabola (see Example 1)	X-axis	$y^2 + z^2 = 4\,ax$	Paraboloid of revolution.
(7) $y^2 - m^2x^2 = 0$, Pair of intersecting lines.	X-axis	$m^2x^2 - y^2 - z^2 = 0$	Circular cone.
(8) $y^2 - m^2 = 0$, Pair of parallel lines.	X-axis	$y^2 + z^2 - m^2 = 0$	Circular cylinder.

3. Suppose that we wish to determine the equation of the surface obtained when the line $2\,x + 6\,y - 3\,z + 1 = 0,\ 3\,x - y + 2\,z - 3 = 0$ revolves about the

line $2x + 6y - 3z + 1 = 0$, $x + y + z = 0$. Evidently the two lines lie in the plane $2x + 6y - 3z + 1 = 0$. Therefore we determine a transformation of coördinates to a new frame of reference $O_1\text{-}X_1Y_1Z_1$, in which this plane becomes the X_1Y_1-plane and the line in which it meets the plane $x+y+z=0$, that is, the axis of revolution, becomes the X_1-axis. As origin O_1 of the new frame, we shall take the point in which the two given lines meet. In the notation of Section 66, we have therefore $a = 1$, $b = -\frac{2}{3}$, $c = -\frac{1}{3}$. Since the X_1-axis is the intersection of the planes $2x + 6y - 3z + 1 = 0$ and $x + y + z = 0$, we find, by use of Theorem 17, Chapter IV (see Section 47, page 87), $\lambda_1 : \mu_1 : \nu_1 = 9 : -5 : -4$; and therefore $\lambda_1 = \dfrac{9}{\sqrt{122}}$, $\mu_1 = -\dfrac{5}{\sqrt{122}}$, $\nu_1 = -\dfrac{4}{\sqrt{122}}$. The Z_1-axis is perpendicular to the plane $2x + 6y - 3z + 1 = 0$; its direction cosines are therefore proportional to 2, 6, -3 (see Theorem 7, Chapter IV, Section 44, page 78). Therefore $\lambda_3 = \frac{2}{7}$, $\mu_3 = \frac{6}{7}$, $\nu_3 = -\frac{3}{7}$. The direction cosines λ_2, μ_2, ν_2 of the Y_1-axis must therefore satisfy the conditions $9\lambda_2 - 5\mu_2 - 4\nu_2 = 0$ and $2\lambda_2 + 6\mu_2 - 3\nu_2 = 0$ (see Corollary 2 of Theorem 13, Chapter III, Section 36, page 64). From this we find, by use of Theorem 4, Chapter II (Section 25, page 41) that $\lambda_2 : \mu_2 : \nu_2 = 39 : 19 : 64$, and that $\lambda_2 = \dfrac{39}{7\sqrt{122}}$, $\mu_2 = \dfrac{19}{7\sqrt{122}}$, $\nu_2 = \dfrac{64}{7\sqrt{122}}$. Hence the equations of transformation are, in view of Theorem 8, Chapter V (Section 66, page 126)

$$\begin{aligned}
x &= \frac{9x_1}{\sqrt{122}} + \frac{39y_1}{7\sqrt{122}} + \frac{2z_1}{7} + 1, &\text{or}\quad x_1 &= \frac{9(x-1) - 5(y+\frac{2}{3}) - 4(z+\frac{1}{3})}{\sqrt{122}},\\
y &= -\frac{5x_1}{\sqrt{122}} + \frac{19y_1}{7\sqrt{122}} + \frac{6z_1}{7} - \frac{2}{3} & y_1 &= \frac{39(x-1) + 19(y+\frac{2}{3}) + 64(z+\frac{1}{3})}{7\sqrt{122}},\\
z &= -\frac{4x_1}{\sqrt{122}} + \frac{64y_1}{7\sqrt{122}} - \frac{3z_1}{7} - \frac{1}{3}, & z_1 &= \frac{2(x-1) + 6(y+\frac{2}{3}) - 3(z+\frac{1}{3})}{7}.
\end{aligned}$$

The first of these two sets of formulas will carry the equations

$$\begin{aligned}
2x + 6y - 3z + 1 &= 0 &\text{over into}\quad 7z_1 &= 0\\
x + y + z &= 0 & \sqrt{122}\,y_1 + 5z_1 &= 0\\
3x - y + 2z - 3 &= 0 & 84x_1 + 113y_1 - 3\sqrt{122}\,z_1 &= 0.
\end{aligned}$$

Hence the equations of the given line may be written in the form $z_1 = 0$, $84x_1 + 113y_1 = 0$; and the equations of the axis of revolution can be put in the form $z_1 = 0$, $y_1 = 0$ (compare Remark 2, following Theorem 18, Chapter IV, Section 49, page 92). The equation in $O_1\text{-}X_1Y_1Z_1$ of the required surface of revolution is therefore obtained by rationalizing the equation $84x_1 + 113\sqrt{y_1^2 + z_1^2} = 0$. This gives the equation $-84^2x_1^2 + 113^2(y_1^2+z_1^2) = 0$. To obtain the equation of the surface in $O\text{-}XYZ$ (from elementary Solid Geometry we know that it is a circular cone), we now substitute for x_1, y_1, and z_1 the expressions in terms of x, y, z to which they are equal by the second set of transformation equations. This gives us for the equation of the required cone

$$-49 \cdot 84^2 (9x - 5y - 4z - \tfrac{11}{3})^2 + 113^2 (39x + 19y + 64z - 5)^2 + 122 \cdot 113^2 (2x + 6y - 3z + 1)^2 = 0.$$

69. Exercises.

1. Prove that, if $z = 0$, $g(x, y) = 0$ are the equations of a curve in the XY-plane, then: (*a*) $g(x, \sqrt{y^2 + z^2}) = 0$ is the equation of the surface of revolution obtained by revolving this curve about the X-axis; (*b*) $g(\sqrt{x^2 + z^2}, y) = 0$ is the equation of the surface of revolution obtained by revolving the curve about the Y-axis.

2. Determine the equation of the surface which is generated when the circle $x = 0$, $y^2 + (z - 5)^2 = 9$ in the YZ-plane, is revolved, (*a*) about the Z-axis; (*b*) about the Y-axis.

NOTE. The surface generated when a circle revolves about a line in its plane not through its center, is called an **anchor ring,** or a **torus.** Which of the surfaces described in this problem is a torus?

3. Determine the equation of the surface obtained by revolving the lemniscate $(x^2 + y^2)^2 = 8\,(x^2 - y^2)$, (*a*) about the X-axis; (*b*) about the Y-axis.

4. Determine the equation of the circular cylinder which is formed when the line $z = 0$, $3\,x - 4\,y + 2 = 0$ revolves about the line $z = 0$, $3\,x - 4\,y - 8 = 0$.

5. Develop the general equation of a torus. (Take the axis of revolution as one of the coördinate axes.)

6 Determine the equation of the surface generated when the equilateral hyperbola $xy = a^2$ is revolved about one of its asymptotes.

7. Determine the equation of the surface of revolution obtained by revolving (*a*) the parabola $z^2 = 4\,ay$ about the Z-axis; (*b*) the semi-cubical parabola $x^2 = z^3$ about the Z-axis; (*c*) the same curve about the X-axis; (*d*) the curve $y = \sin x$ about the X-axis.

8. Determine the equation of the oblate spheroid obtained by revolving the ellipse $\dfrac{(x-3)^2}{4} + \dfrac{(y-2)^2}{9} = 1$, $z = 0$ about the line $y = 2$, $z = 0$.

9. Determine the equation of the circular cone which is generated when the line $8\,x - 4\,y + z - 2 = 0$, $2\,x - y - 2\,z + 3 = 0$ is revolved about the line $8\,x - 4\,y + z - 2 = 0$, $x + 2\,y + 2\,z - 4 = 0$.

10. The hyperbola $(x - 4)^2 - \dfrac{(z-2)^2}{4} = 1$, $x = 0$ is revolved about its asymptote with positive slope. Determine the equation of the surface of revolution which is generated.

70. The Shape of a Surface Determined from its Equation. Contour Lines. One of the fundamental problems of Solid Analytical Geometry is that of forming a clear picture of the shape of the surface which is the locus of a given equation. We have already obtained a number of partial solutions of this problem; we will begin by summarizing these:

(*a*) the locus of an equation of the first degree is a plane.

(*b*) the locus of an equation from which one variable is absent is a cylindrical surface.

(*c*) the locus of an equation which can be written in one of the forms $f(x, \sqrt{y^2 + z^2}) = 0$, $f(y, \sqrt{z^2 + x^2}) = 0$ or $f(z, \sqrt{x^2 + y^2}) = 0$ is a surface of revolution; the meridian curve of such a surface can then be found by the methods of Plane Analytical Geometry.

To these we now add the following further results.

THEOREM 2. **The locus of an equation which can be reduced to the form $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ is a sphere whose center is at (a, b, c) and whose radius is r.**

The proof of this theorem is left to the reader.

DEFINITION II. **A *conical surface* is a surface generated by a line, extending indefinitely, which moves in such a way as to pass always through a fixed point, called the *vertex*, and successively through the points of a fixed curve, called the *directrix*.**

Remark. It is a consequence of this definition that a conical surface consists of two parts, one on each side of the vertex and connected at the vertex; these parts are called the "**sheets**" or "**nappes**" of the surface (compare Section 68, Example 2, part 4, page 133). If the directrix is a plane curve and the vertex lies in the plane of this curve, the conical surface is a sector of this plane; if the directrix is a straight line, we obtain the entire plane. In case the directrix consists of a pair of lines which do not both lie in a plane with the vertex, the surface consists of a pair of intersecting planes; if both lines lie in a plane with the vertex, the surface reduces to a single plane counted twice. According as the directrix is a curve of the first, second, third order, etc., the conical surface is said to be of the first, second, third order, etc.; compare also Remark 1, Section 82, page 166.

THEOREM 3. **The locus of an equation in x, y, and z is a conical surface with vertex at the origin if and only if the equation is homogeneous.**

Proof. Recalling the definition of a homogeneous equation (Section 20, page 35) we observe that if the equation $f(x, y, z) = 0$ is a homogeneous equation of degree n, then and only then will all the terms in the polynomial $f(x, y, z)$ be of the nth degree in the three variables jointly, and hence we know that $f(kx, ky, kz) = k^n f(x, y, z)$. Suppose now that $P(\alpha, \beta, \gamma)$ is a point on the locus of the equation $f(x, y, z) = 0$, supposed to be homogeneous of degree n.

The coördinates of an arbitrary point on the line OP are given by the equations $x = t\alpha$, $y = t\beta$, $z = t\gamma$, in which t is a parameter that varies from point to point along the line (see Corollary 3 of Theorem 6, Chapter III, Section 33, page 56). And it follows from the homogeneity of the equation that $f(t\alpha, t\beta, t\gamma) = t^n f(\alpha, \beta, \gamma) = 0$. Consequently the entire line OP lies on the surface represented by the equation $f(x, y, z) = 0$, if P does. The line OP will therefore generate the surface if P passes successively through the points of a curve in which the surface is cut by a plane not through the origin. The proof of the converse of this theorem is left to the reader.

Remark 1. The locus of the linear homogeneous equation $ax + by + cz = 0$ is a plane through the origin. In view of the remark preceding Theorem 3 this is a special case of a conical surface

Remark 2. If an equation is homogeneous in $x - a$, $y - b$, $z - c$, translation of the axes to the point $A(a, b, c)$ as a new origin will reduce it to an equation which is homogeneous in $x' = x - a$, $y' = y - b$, $z' = z - c$ (see Theorem 2, Chapter V, Section 61, page 115). Its locus is therefore a conical surface whose vertex is at $A(a, b, c)$. Conversely, translation of axes to A as origin puts the vertex of a conical surface with vertex at A at the origin of the new reference frame. We can therefore state the following corollary:

COROLLARY. **The locus of an equation in x, y, z is a conical surface with vertex at the point (a, b, c) if and only if it is homogeneous in $x - a$, $y - b$, and $z - c$.**

THEOREM 4. **The locus of the equation $f(x, y, z) = 0$ is symmetric with respect to the YZ-plane* if and only if the equations $f(-x, y, z) = 0$ and $f(x, y, z) = 0$ are equivalent;† the locus of this equation is symmetric with respect to the Z-axis* if and only if the equations**

* Two points A and B are said to be **symmetric with respect to a plane** (or **line**) if the segment AB is bisected perpendicularly by the plane (or line); they are called **symmetric with respect to a point** if the segment AB is bisected by that point. A surface is said to be symmetric with respect to plane (line, point) if with every surface point A there is associated a surface point B such that A and B are symmetric with respect to the plane (line, point).

† Two equations are called **equivalent** if any set of values of the variables which satisfies either of the equations, also satisfies the other equation; e.g., the equations $x^2 - 2y^3 + 4z^2 + y = 0$ and $(-x)^2 - 2y^3 + 4(-z)^2 + y = 0$ are equivalent, also the equations $x - 3y + 2z^3 = 0$ and $-2x + 6y - 4z^3 = 0$.

$f(-x, -y, z) = 0$ and $f(x, y, z) = 0$ are equivalent; this locus is symmetric with respect to the origin if and only if the equations $f(-x, -y, -z) = 0$ and $f(x, y, z) = 0$ are equivalent.

This theorem and its obvious counterparts which assert symmetry with respect to the other coördinate planes and coördinate axes are immediate extensions of well-known theorems of Plane Analytical Geometry. The reader is entrusted with proving them. We see, for example, that the locus of the equation $x^2 - 2y^3 + 4z^2 + y = 0$ is symmetric with respect to the Y-axis, and that the locus of the equation $x - 3y + 2z^3 = 0$ is symmetric with respect to the origin. Compare Exercise 3, Section 31, page 52.

Of great value in determining the shape of a surface represented by an equation are the contour lines of the surfaces; these are the projections upon a fixed plane of the intersections of the surface with a series of planes parallel to this fixed plane. We shall make use only of sets of planes which are parallel to the coördinate planes; accordingly we lay down the following definition:

Definition III. **The *X-contour lines* of a surface are the projections on the YZ-plane of the curves in which the surface is met by planes parallel to the YZ-plane, that is, by the planes whose equations are $x = a$; the *Y-contour lines* (*Z-contour lines*) are the projections on the ZX-plane (the XY-plane) of the curves in which the surface is met by the planes parallel to the ZX-plane (the XY-plane), that is, by the planes $y =$ constant (the planes $z =$ constant).**

It follows immediately from this definition, in conjunction with Theorem 2, Chapter IV (Section 40, page 71), that the X-contour lines of the surface $f(x, y, z) = 0$ are the curves in the YZ-plane whose plane equations are $f(k, y, z) = 0$, in which k is a parameter taking all real values. Similarly the Y-contour lines and the Z-contour lines are given by the equations $f(x, k, z) = 0$ and $f(x, y, k) = 0$, respectively.

If we plot a set of contour lines, affixing to each of them the value of the parameter k to which it corresponds, we obtain a diagram which we shall call a **contour map** of the surface; such contour maps should be of considerable aid in forming a mental picture of the locus of an equation. The contour maps published by the U. S. Geological Survey are approximately Z-contour maps of portions of the earth's surface, if the level plane at some point in the region is taken as the XY-plane.

Remark. In engineering practice various other methods are used for representing space configurations in a plane drawing. Of these we mention Descriptive Geometry and Perspective Drawing. These subjects are treated in books especially devoted to them. We shall not attempt to discuss them here; the interested reader will have little difficulty in getting access to such books.

Example. To determine the X-contour lines of the locus of the equation $\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{16} = 1$, we consider the set of curves in the YZ-plane whose plane equations are $\frac{k^2}{4} - \frac{y^2}{9} + \frac{z^2}{16} = 1$. There are three cases to be distinguished:

(*a*) $|k| < 2$, that is, $-2 < k < 2$. In this case $k^2 < 4$, so that $1 - \frac{k^2}{4} > 0$. The equations of the X-contour lines for these values of the parameter k can therefore be put in the form

$$\frac{z^2}{16\left(1 - \frac{k^2}{4}\right)} - \frac{y^2}{9\left(1 - \frac{k^2}{4}\right)} = 1.$$

From this we conclude that these contour lines are hyperbolas whose center is at the origin; that the transverse axis is along the Z-axis and the conjugate axis along the Y-axis. The semi-axes of the hyperbola are $a = 2\sqrt{4 - k^2}$ and $b = \frac{3\sqrt{4 - k^2}}{2}$; hence $\frac{a}{b} = \frac{4}{3}$ and all the hyperbolas of the set have the same asymptotes, namely, the lines $z = \pm\frac{4y}{3}$, and the same eccentricity, $\frac{5}{4}$. The foci are on the Z-axis at the points $\left(0, 0, \pm\frac{5\sqrt{4 - k^2}}{2}\right)$.

(*b*) $|k| = 2$, that is, $k = \pm 2$. The equation of the contour lines now reduces to $-\frac{y^2}{9} + \frac{z^2}{16} = 0$; the locus of this equation consists of a pair of intersecting lines, namely the asymptotes common to the hyperbolas discussed under (*a*).

(*c*) $|k| > 2$, that is, $k > 2$, or $k < -2$. In this case $k^2 > 4$, so that $1 - \frac{k^2}{4} < 0$ and the equation takes the form $\frac{y^2}{9\left(\frac{k^2}{4} - 1\right)} - \frac{z^2}{16\left(\frac{k^2}{4} - 1\right)} = 1.$

The contour lines are now hyperbolas with center at O, the transverse axis along the Y-axis and conjugate axis along the Z-axis. The semi-axes are $a = \frac{3\sqrt{k^2 - 4}}{2}$ and $b = 2\sqrt{k^2 - 4}$; the foci are at the points $\left(0, \pm\frac{5\sqrt{k^2 - 4}}{2}, 0\right)$ on the Y-axis; the asymptotes are the same as those of the hyperbolas in (*a*), namely, the lines $z = \pm\frac{4y}{3}$, and the eccentricity is again equal to $\frac{5}{4}$.

The X-contour map of this surface is sketched in Fig. 22, the plane of the drawing being the ZY-plane. It follows from Theorem 4 that the surface is symmetric with respect to the three coördinate planes, the three coördinate axes and the origin. The part of the surface suggested by this contour map, which lies to the right of the ZY-plane is therefore duplicated by a symmetric part to the left of this plane.

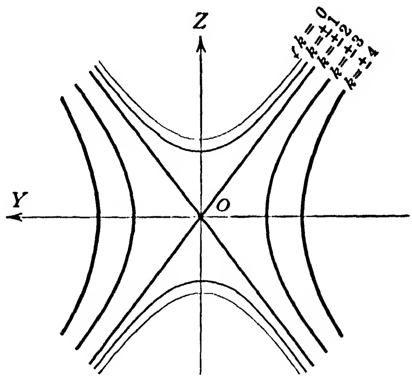

FIG. 22

It should be easy for the reader to show that the Z-contour lines of this surface are also hyperbolas and that the Z-contour map has the same general character as in Fig. 22. The Y-contour lines are ellipses; the reader should construct the Y-contour map. The surface represented by this equation is called a **hyperboloid of one sheet.** Of the axes of symmetry, the X- and Z-axes meet the surface in real points; they are called the **transverse axes** of the surface. The Y-axis does not meet the surface in real points; it is called the **conjugate axis** of the surface.

71. Some Facts from Plane Analytical Geometry. In preparation for the construction of the contour maps of other surfaces, we summarize in this section some facts from Plane Analytical Geometry with which the reader should be familiar.

(1) the locus of the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a > b$, is an ellipse, whose major and minor axes are along the X- and Y-axis respectively. The center is at the origin; the vertices at the points $(\pm a, 0)$ and the foci at the points $(\pm c, 0)$, where $c^2 = a^2 - b^2$. The eccentricity e is equal to $\dfrac{c}{a}$, which is <1; the directrices are the lines $x = \pm \dfrac{a}{e}$. The ratio of the distances of any point on the ellipse from a focus and from the corresponding directrix is equal to e; the sum of the distances of any point on the ellipse from the two foci equals $2\,a$.

The locus of the equation $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, $a > b$, is obtained from

the ellipse just discussed by interchanging the rôles of the X- and Y-axes.

(2) The locus of the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is an hyperbola with center at the origin, transverse axis along the X-axis and conjugate axis along the Y-axis. The vertices are the points $(\pm a, 0)$, the foci at $(\pm c, 0)$, where $c^2 = a^2 + b^2$. The eccentricity e is equal to $\frac{c}{a}$, which is >1; the directrices are the lines $x = \pm \frac{a}{e}\cdot$ The asymptotes are the lines $\frac{x}{a} \pm \frac{y}{b} = 0$. The ratio of the distances of any point on the hyperbola from a focus and from the corresponding directrix is equal to e; the difference between the distances of any point on the hyperbola from the two foci is equal to $\pm 2a$.

The properties of the locus of the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ are analagous to those of the curve just discussed. The vertices and foci lie on the Y-axis, which is therefore the transverse axis; the X-axis is the conjugate axis. The asymptotes, the number c and the eccentricity are the same as the corresponding elements of the first hyperbola. The two curves are called **conjugate hyperbolas.**

(3) The locus of the equation $y^2 = 4p(x - a)$ is a parabola whose axis is on the X-axis. The vertex is at the point $(a, 0)$; the focus at the point $(a + p, 0)$. The curve will therefore open in the direction of the positive or the negative X-axis, according as p is positive or negative. The directrix is the line $x = a - p$. The ratio of the distances of any point on the parabola from the focus and the directrix is equal to 1.

72. Some Special Surfaces. We should now be able to construct readily the contour maps of the loci of a number of special equations. The work will only be sketched; the reader should work out carefully the details of the constructions.

(*a*) the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; $a > b > c$.*

* The case in which two or more of the numbers a, b, c are equal has been discussed in Example 2, Section 68 (page 133).

The Z-contour lines are given by the equation $\dfrac{x^2}{a^2\left(1 - \dfrac{k^2}{c^2}\right)} + \dfrac{y^2}{b^2\left(1 - \dfrac{k^2}{c^2}\right)} = 1$. They are ellipses whose semi-axes are equal to $\dfrac{a\sqrt{c^2 - k^2}}{c}$ and $\dfrac{b\sqrt{c^2 - k^2}}{c}$. The eccentricity is equal to $\dfrac{\sqrt{a^2 - b^2}\sqrt{c^2 - k^2}}{a\sqrt{c^2 - k^2}} = \dfrac{\sqrt{a^2 - b^2}}{a}$ and therefore independent of k. The foci are at the points $\left(\pm\dfrac{\sqrt{a^2 - b^2}\sqrt{c^2 - k^2}}{c}, 0, 0\right)$. For $k = 0$, the semi-axes have their largest values, a and b; as $|k|$ increases from 0 to c, the semi-axes decrease; for $k = \pm c$, they are 0 and the ellipses shrink down to the points $(0, 0, \pm c)$ as k tends towards $\pm c$; when $|k| > c$, the ellipses are imaginary. We conclude that the contour map consists of a series of similar ellipses. In Fig. 23 the contour lines for the part of the surface above the XY-plane are sketched.

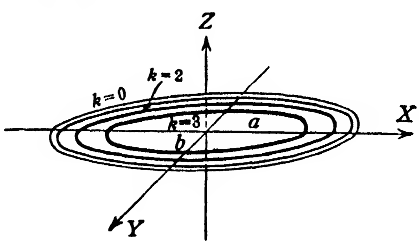

FIG. 23

The surface is symmetric with respect to the XY-plane; the contour map for the part of the surface below the XY-plane is therefore identical with the one here drawn. This surface is called an **ellipsoid**; its shape is suggested in perspective by Fig. 24. The segments a, b, and c are called the **semi-axes** of the surface; they are equal to one half of the segments which the surface cuts off on the axes of symmetry.

(b) the equations $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$; $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$; and $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

An equation of the second of these types has already been discussed in the Example of Section 70. We shall therefore leave the discussion of these equations to the reader. Surfaces represented by an equation of this form are called hyperboloids of one sheet;

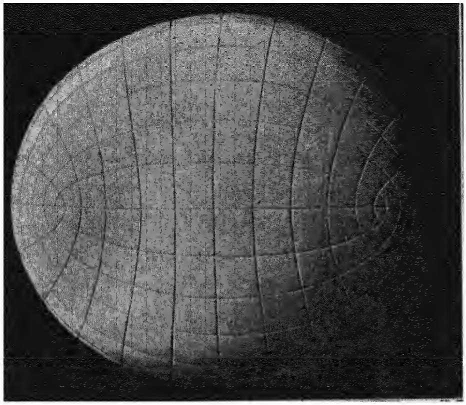

Fig. 24

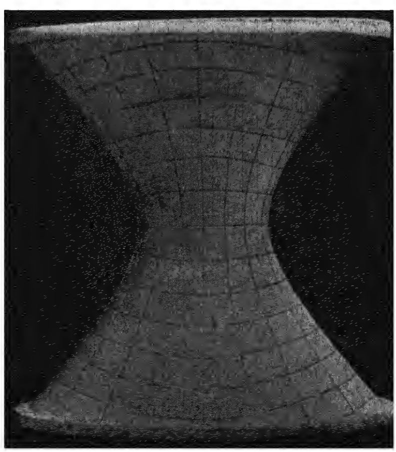

Fig. 25

the segments a, b, and c are its semi-axes. Figure 25 suggests the appearance of an hyperboloid of one sheet.

(c) the equations $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$; $-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$; and $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

The loci of these three equations are of the same general character. Interchanges among the coördinate axes will carry one of them over into the others; we shall therefore make a discussion of the first of these equations only, and we shall suppose that $b > c$.* The X-contour lines are determined by the equation $\frac{y^2}{b^2\left(\frac{k^2}{a^2} - 1\right)} + \frac{z^2}{c^2\left(\frac{k^2}{a^2} - 1\right)} = 1$. They are ellipses, whose semi-axes are equal to $\frac{b\sqrt{k^2 - a^2}}{a}$ and $\frac{c\sqrt{k^2 - a^2}}{a}$; the eccentricity is equal to $\frac{\sqrt{b^2 - c^2}}{b}$ and is therefore independent of k. For $|k| < a$, the semi-axes are not real; therefore no points of the locus are found between the planes $x = a$ and $x = -a$. For $k = \pm a$, the ellipses reduce to the points $(\pm a, 0, 0)$. For $|k| > a$, we obtain a series of similar ellipses, which increase indefinitely in size as $|k|$ increases. The surface consists therefore of two sheets, each of which extends indefinitely, one in the direction of the positive X-axis, the other in the direction of the negative X-axis. The surface is called an **hyperboloid of two sheets** (see Fig. 26).

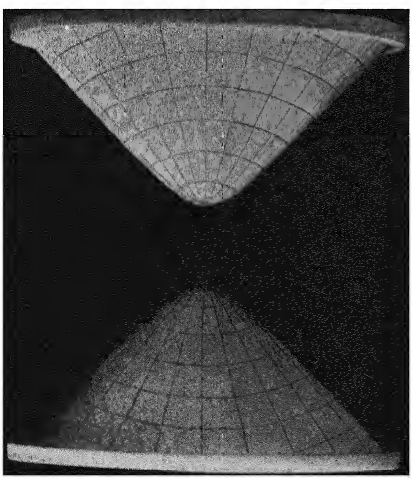

FIG. 26

* In case $b = c$, the locus is a hyperboloid of revolution of two sheets, see Example 2, Section 68, page 133.

Of the three axes of symmetry, the X-axis meets the surface in real points; it is called the **transverse axis.** The other axes of symmetry which do not meet the surface in real points are called the **conjugate axes.**

(*d*) the equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=-1.$

It should be clear that there are no real points whose coördinates satisfy this equation; its locus is called an **imaginary ellipsoid.**

(*e*) the equations $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=0$ and $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=0.$

It follows from Theorem 3 (Section 70, page 136) that the locus of these equations consists of conical surfaces of the second order. One contour map consists of similar ellipses, the other two of

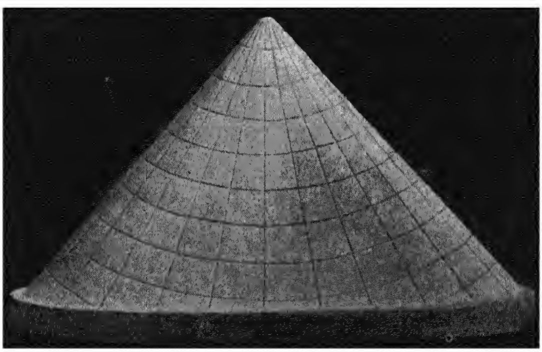

FIG. 27

similar hyperbolas. In case the denominators in the two terms with like signs are equal, the locus is a circular cone (see Example 2, Section 68, page 133); otherwise it is an **elliptic cone** (see Fig. 27).

(*f*) the equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=0.$

The only real point on the locus of this equation is the origin; the surface is called an **imaginary cone.**

(*g*) the equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2\,pz, \quad a>b.$

The Z-contour lines are the similar ellipses $\frac{x^2}{2\,pka^2} + \frac{y^2}{2\,pkb^2} = 1$, whose semi-axes are equal to $a\sqrt{2\,pk}$ and $b\sqrt{2\,pk}$, and whose eccentricity is equal to $\frac{\sqrt{a^2 - b^2}}{a}$. For $k = 0$, the ellipse reduces to the origin. If $p > 0$, the ellipses are real for $k > 0$ and imaginary for $k < 0$; if $p < 0$, the situation is reversed. Consequently the surface lies entirely on one side of the XY-plane and extends indefinitely on that side, on the side of the positive Z-axis if $p > 0$, on the side of the negative Z-axis if $p < 0$. The X-contour lines and the Y-contour lines are parabolas. The surface is called an **elliptic paraboloid** (see Fig. 28). It should be clear that the equations $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 2\,px$ and $\frac{z^2}{c^2} + \frac{x^2}{a^2} = 2\,py$ also represent elliptic paraboloids.

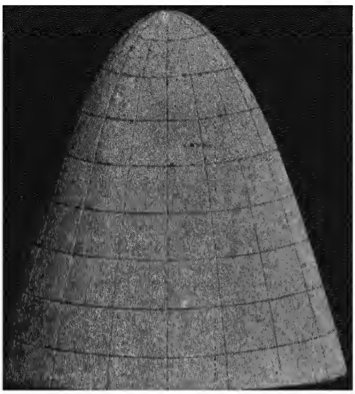

FIG. 28

(*h*) the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2\,pz$.

The X-contour lines are the parabolas in the YZ-plane which are represented by the equation $\frac{y^2}{b^2} = \frac{k^2}{a^2} - 2\,pz$, which can be written in the form $y^2 = -2\,pb^2\left(z - \frac{k^2}{2\,pa^2}\right)$. The axes of these parabolas are along the Z-axis and the vertices are at the points $\left(0, 0, \frac{k^2}{2\,pa^2}\right)$. Therefore, if $p > 0$, the parabolas will all extend in the direction of the negative Z-axis, whereas their vertices will move upward along the Z-axis as k increases. But, if $p < 0$, the parabolas extend in the direction of the positive Z-axis, and their vertices move down-

ward along the Z-axis as k increases. In Fig. 29, the X-contour map has been drawn, for the case $p > 0$, of the part of the surface which lies on the right side of the YZ-plane; since the surface is obviously symmetric with respect to the YZ-plane, the X-contour map of the other part of the surface is identical with the one drawn in this figure.

The Y-contour map consists of the parabolas

$$x^2 = 2\,pa^2\left(z + \frac{k^2}{2\,pb^2}\right).$$

It should be clear that, if $p > 0$, the contour lines are upward extending parabolas which move downward as k increases; and if $p < 0$, downward extending parabolas which move upward as k increases.

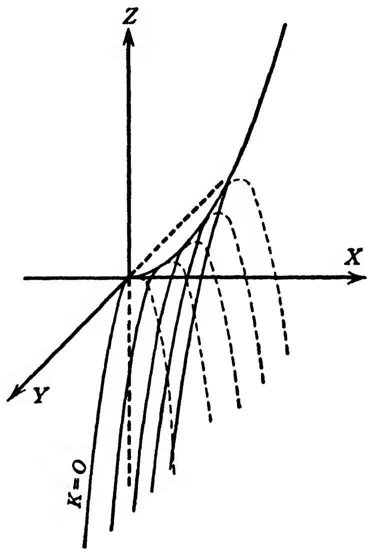

FIG. 29

The Z-contour lines are hyperbolas. The surface is called an

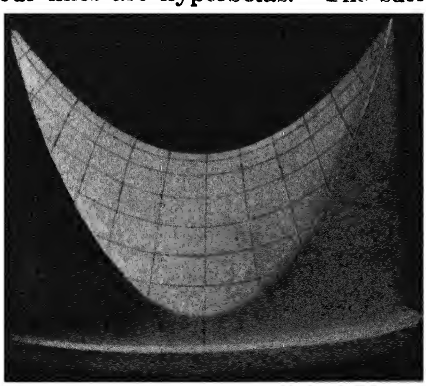
FIG. 30

hyperbolic paraboloid (see Fig. 30). It should be clear that the equations $\dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 2\,px$ and $\dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} = 2\,py$ also represent hyperbolic paraboloids.

73. Exercises.

1. Show that the locus of each of the following equations is a sphere; determine for each of them the radius and the coördinates of the center:

(a) $x^2 + y^2 + z^2 + 4x - 6y + 4z - 8 = 0$;
(b) $x^2 + y^2 + z^2 - 3x + 4y - 2z + 3 = 0$;
(c) $x^2 + y^2 + z^2 + 6x + 4y + 4z + 17 = 0$;
(d) $x^2 + y^2 + z^2 - 8x - 6y - 4z + 30 = 0$;
(e) $x^2 + y^2 + z^2 + 2ax + 2by + 2cz + d = 0$.

2. Prove that the locus of the pair of equations $2x - 6y + 3z - 5 = 0$ and $x^2 + y^2 + z^2 - 4x - 6y + 8z - 7 = 0$ is a circle. Determine the radius of this circle and the coördinates of its center.

Hint: The center of the circle is the foot of the perpendicular dropped on the plane from the center of the sphere; the length of this perpendicular, the radius of the circle and the radius of the sphere form a right triangle.

3. Show that the points common to the two spheres $x^2 + y^2 + z^2 - 6x + 6y - 8z + 3 = 0$ and $x^2 + y^2 + z^2 - 3x + 2y - 5z + 4 = 0$ lie in a plane.

Hint: Write the equation of the pencil of spheres through the points common to the two given spheres and show that the pencil contains a plane. (Compare Remark 4, after Theorem 18, Chapter IV, Section 49, page 92.)

4. Show that the pencil of spheres

$$k_1(x^2 + y^2 + z^2 + 2a_1x + 2b_1y + 2c_1z + d_1) + k_2(x^2 + y^2 + z^2 + 2a_2x + 2b_2y + 2c_2z + d_2) = 0$$

contains a plane and prove that this plane is perpendicular to the line which joins the centers of the two spheres, given by $k_1 = 0$, $k_2 = 1$ and $k_1 = 1$, $k_2 = 0$.

5. Prove that the curve determined by the equations $x - y + z = 0$ and $x^3 + y^3 = z^3$ is symmetric with respect to the origin but not symmetric with respect to any of the coördinate axes or coördinate planes.

6. Set up the conditions under which the plane $ax + by + cz + d = 0$ meets the sphere $x^2 + y^2 + z^2 + 2a_1x + 2b_1y + 2c_1z + d_1 = 0$ in real points.

7. Determine, supposing that the conditions of Exercise 6 are satisfied, the radius and the coördinates of the center of the circle in which the plane and the sphere meet.

8. Determine the equation of the conical surface of the second order whose vertex is at the origin and whose directrix is the circle $2x - y - 3z + 7 = 0$, $x^2 + y^2 + z^2 - 2x - 4y + 2z - 5 = 0$.

Hint: Two methods for solving this problem should be considered: First method — The required equation must be homogeneous of the second degree; moreover it must be of the general form $k_1S_1 + k_2S_2 = 0$, where $S_1 = 0$ and $S_2 = 0$ represent respectively the plane and the sphere, where k_1 is a linear function of x, y, and z, and where k_2 is a constant. Second method — If

$P(x, y, z)$ is an arbitrary point on the conical surface with vertex at the origin, then there must exist a factor of proportionality r, such that rx, ry, rz satisfy the equations of the plane and the sphere. Elimination of r leads to the required equation.

9. Find the equation of the circular conical surface whose vertex is at $V(-3, 2, -1)$ and whose directrix is the circle $3x + y - 2z + 4 = 0$, $x^2 + y^2 + z^2 + 4x - 6y - 4z - 7 = 0$.

10. Determine the equation of the conical surface with vertex at the origin whose directrix is the curve of intersection of the plane $x + 2y - 2z - 4 = 0$ and the ellipsoid $\frac{x^2}{4} + \frac{y^2}{6} + \frac{z^2}{2} = 1$.

11. Write the equation of the circular conical surface whose vertex is at $V(\alpha, \beta, \gamma)$ and whose directrix is the curve of intersection of the plane $ax + by + cz + d = 0$ and the sphere $x^2 + y^2 + z^2 + 2a_1x + 2b_1y + 2c_1z + d_1 = 0$.

12. Construct

(*a*) the Y-contour map of the locus of the equation $xy = 2z$.
(*b*) the Z-contour map of the same surface.
(*c*) the X-contour map of the surface $xy + yz + zx = 0$.

74. The Intersections of a Surface and a Line. To find the points in which a line meets a surface, we have to solve simultaneously three equations; two of these equations are linear, namely, the equations of the line, the third may be of any degree. The natural way to attack this algebraic problem would probably seem to be to solve the linear equations for two of the variables in terms of the third, to substitute these values in the third equation and to solve the resulting equation for the third variable. For example, if we wished to determine the points in which the line $2x - y + z - 4 = 0$, $x + 2y - 3z + 2 = 0$ meets the surface $x^2 - 2xz + 7y = 0$, we would derive from the first two equations that $z = 5x - 6$ and $y = 7x - 10$; substitution in the third equation would then lead to the equation $-9x^2 + 61x - 70 = 0$. If this equation were solved for x, the coördinates of the desired points could readily be found.

This method, although frequently useful in special cases, lacks symmetry in the treatment of the variables, because it involves the selection of one variable (in the example above this was x), in terms of which the others are expressed; it casts the three variables in non-interchangeable rôles. It has been found that if symmetry is maintained, greater elegance and clarity is introduced in the treatment of algebraic problems. Without going any further into a discussion of mathematical esthetics, to which

we are led quite naturally at this point, we shall proceed with the problem of finding the intersections of a surface and a line in a more symmetric form.

For this purpose, we take the equations of the line in the parametric form of Theorems 15 or 16, Chapter IV (Section 47, page 86). Let the equations of the line be

$$x = \alpha + \lambda s, \quad y = \beta + \mu s, \quad z = \gamma + \nu s$$

and let the surface be represented by the equation $f(x, y, z) = 0$, where $f(x, y, z)$ represents a polynomial of degree n in x, y, and z. To determine the points common to the line and the surface we have now four equations in the four variables x, y, z and s. To solve them, we substitute the expressions for x, y, and z given by the equations of the line in the equation of the surface. This leads to the equation

$$(1) \quad f(\alpha + \lambda s, \quad \beta + \mu s, \quad \gamma + \nu s) = 0,$$

which has to be solved for s. When this solution has been accomplished, the coördinates of the required point can be determined at once. If we treat the example of the preceding paragraph by this method, we derive first the parametric equations of the line (see the Examples in Section 47, pages 88 and 89); using the form of Theorem 16, Chapter IV, we find for them

$$x = 2 + t, \quad y = 4 + 7\,t, \quad z = 4 + 5\,t.^*$$

Substitution of these values of x, y, and z in the equation of the surface gives the quadratic equation $9\,t^2 - 25\,t - 16 = 0$. From this equation we obtain $t = \dfrac{25 \pm \sqrt{1201}}{18}$ and hence $x_1 = \dfrac{61 + \sqrt{1201}}{18}$, $y_1 = \dfrac{247 + 7\sqrt{1201}}{18}$, $z_1 = \dfrac{197 + 5\sqrt{1201}}{18}$; $x_2 = \dfrac{61 - \sqrt{1201}}{18}$, $y_2 = \dfrac{247 - 7\sqrt{1201}}{18}$, $z_2 = \dfrac{197 - 5\sqrt{1201}}{18}$. The points of inter-

* If we put $t = \dfrac{s}{\sqrt{75}}$, we obtain the parametric equations in the s-form of Theorem 15 (see Corollaries 2 and 3 of Theorem 10, Chapter III, Section 34, page 60). In numerical problems, the t-form of the parametric equations is usually the more convenient; for theoretical discussions the s-form is usually to be preferred.

section of the line and the surface are approximately at $P_1(5.3, 27.2, 20.6)$ and $P_2(1.5, .2, 1.3)$.

75. Digression on Taylor's Theorem. The discussion of the solution of equation (1) of the preceding section in the case of a general polynomial, can be made in a very direct way by means of Taylor's theorem,* which takes on a particularly simple form for polynomials. This theorem tells us that if $F(x, y, z)$ is a polynomial, then

$$F(a+h, b+k, c+l) = F(a, b, c) + [hF_1(a, b, c) + kF_2(a, b, c) + lF_3(a, b, c)] + \frac{1}{2!}\cdot[h^2F_{11}(a, b, c) + k^2F_{22}(a, b, c) + l^2F_{33}(a, b, c) + 2\,klF_{23}(a, b, c) + 2\,lhF_{31}(a, b, c) + 2\,hkF_{12}(a, b, c)] + \cdots + \frac{1}{n!}[hF_1(a, b, c) + kF_2(a, b, c) + lF_3(a, b, c)]^{(n)}.$$

In this formula $F_1(a, b, c)$, $F_2(a, b, c)$ and $F_3(a, b, c)$ are abbreviations for the partial derivatives $\left.\frac{\partial F}{\partial x}\right|_{x=a,\ y=b,\ z=c}$, $\left.\frac{\partial F}{\partial y}\right|_{x=a,\ y=b,\ z=c}$, $\left.\frac{\partial F}{\partial z}\right|_{x=a,\ y=b,\ z=c}$ respectively; $F_{11}(a, b, c)$, F_{22}, F_{33}, F_{23}, F_{31}, F_{12} represent the second partial derivatives, for example, $F_{12}(a, b, c) = \left.\frac{\partial^2 F}{\partial x \partial y}\right|_{x=a,\ y=b,\ z=c}$; similarly $F_{312}(a, b, c)$ will be used to designate $\left.\frac{\partial^3 F}{\partial z \partial x \partial y}\right|_{x=a,\ y=b,\ z=c}$.†

We observe that the expression in the second pair of brackets in the formula closely resembles the square of that in the first pair of brackets, which involves only the first derivatives of F. For we see that $[hF_1(a, b, c) + kF_2(a, b, c) + lF_3(a, b, c)]^2 = h^2F_1^2(a, b, c) + k^2F_2^2(a, b, c) + l^2F_3^2(a, b, c) + 2\,klF_2(a, b, c)\,F_3(a, b, c) + 2\,lhF_3(a, b, c)F_1(a, b, c) + 2\,hkF_1(a, b, c)F_2(a, b, c)$. The second order terms are obtained from this square if we replace F_1^2 by F_{11}, F_1F_2 by F_{12}, etc., where these symbols designate second partial derivatives in accordance with the notation explained above.

* For a fuller discussion of this important theorem the reader is referred to books on the Calculus.

† Throughout our further work we shall use for partial derivatives the subscript notation which has here been introduced for the arbitrary function $F(x, y, z)$.

We shall therefore denote the set of second order terms by the abbreviated notation $[hF_1(a, b, c) + kF_2(a, b, c) + lF_3(a, b, c)]^{(2)}$; and we shall call this a "*symbolic square.*" Now the further terms on the right side of the formula which states Taylor's theorem for polynomials are the "*symbolic cube,*" the "*symbolic fourth power,*" etc., to the "*symbolic n-th power*" of $hF_1(a, b, c) + kF_2(a, b, c) + lF_3(a, b, c)$, divided by $3!$, $4!$ etc., $n!$ respectively. They are obtained from the ordinary cube, fourth power, to nth power of this expression, if products of first partial derivatives are replaced by corresponding higher partial derivatives; for example, $F_1{}^2F_3F_2$ should be replaced by $\dfrac{\partial^4 F}{\partial x^2 \partial z \partial y}$, etc.

Remark. We notice that each of these symbolic powers are homogeneous functions of h, k, and l.

Let us consider as an example the function

$$F(x, y, z) = x^3 - 3x^2y + 2yz^2 + z^2 - 5x + 3y - 4.$$

Here

$$F_1 = 3x^2 - 6xy - 5,\ F_2 = -3x^2 + 2z^2 + 3,\ F_3 = 4yz + 2z;$$
$$F_{11} = 6x - 6y,\ F_{22} = 0,\ F_{33} = 4y + 2,\ F_{23} = 4z,\ F_{31} = 0,\ F_{12} = -6x;\ F_{111} = 6,\ F_{222} = F_{333} = 0,\ F_{112} = -6,$$
$$F_{113} = F_{221} = F_{223} = F_{331} = F_{231} = 0,\ F_{332} = 4.\quad \text{Hence}$$
$$hF_1 + kF_2 + lF_3 = (3x^2 - 6xy - 5)h + (-3x^2 + 2z^2 + 3)k + (4yz + 2z)l,$$
$$(hF_1 + kF_2 + lF_3)^{(2)} = (6x - 6y)h^2 + (4y + 2)l^2 + 8zkl - 12xhk,$$
$$(hF_1 + kF_2 + lF_3)^{(3)} = 6h^3 - 18h^2k + 12l^2k.$$

We conclude therefore that

$$(a + h)^3 - 3(a + h)^2(b + k) + 2(b + k)(c + l)^2 + (c + l)^2 - 5(a + h) + 3(b + k) - 4 = a^3 - 3a^2b + 2bc^2 + c^2 + 3b - 5a - 4 + [(3a^2 - 6ab - 5)h + (-3a^2 + 2c^2 + 3)k + (4bc + 2c)l] + [(3a - 3b)h^2 + (2b + 1)l^2 + 4ckl - 6ahk] + h^3 - 3h^2k + 2l^2k.$$

76. The Intersections of a Surface and a Line, *continued.* We are now prepared for a further discussion of equation (1) of Section 74 (see page 150). If we apply Taylor's theorem to the left-hand member of this equation, it takes the following form:

$$f(\alpha, \beta, \gamma) + [\lambda s f_1(\alpha, \beta, \gamma) + \mu s f_2(\alpha, \beta, \gamma) + \nu s f_3(\alpha, \beta, \gamma)] + \frac{1}{2!} \times$$

$$[\lambda s f_1 + \mu s f_2 + \nu s f_3]^{(2)} + \cdots + \frac{1}{n!} \cdot [\lambda s f_1 + \mu s f_2 + \nu s f_3]^{(n)} = 0.$$

But since each of the symbolic powers is a homogeneous function (see Remark in Section 75), this equation reduces to an algebraic equation in s of degree n:

$$f(\alpha, \beta, \gamma) + s[\lambda f_1 + \mu f_2 + \nu f_3]_{\alpha, \beta, \gamma} + \frac{s^2}{2!} \cdot [\lambda f_1 + \mu f_2 + \nu f_3]^{(2)}{}_{\alpha, \beta, \gamma}$$

$$+ \cdots + \frac{s^n}{n!} \cdot [\lambda f_1 + \mu f_2 + \nu f_3]^{(n)}{}_{\alpha, \beta, \gamma} = 0,$$

where the notation α, β, γ in the subscript position after a bracket serves to indicate that α, β, and γ are to be substituted for the variables x, y, and z respectively within this bracket. We shall state our result in the following form:

THEOREM 5. **The parameter values of the points in which the locus of the equation $f(x, y, z) = 0$, whose left-hand side is a polynomial, is met by the line $x = \alpha + \lambda s$, $y = \beta + \mu s$, $z = \gamma + \nu s$, are the roots of the algebraic equation**

$$(1)\quad a_0 s^n + a_1 s^{n-1} + \cdots + a_k s^{n-k} + \cdots + a_{n-1} s + a_n = 0,$$

where

$$(2)\quad a_n = f(\alpha, \beta, \gamma),\ a_{n-1} = [\lambda f_1 + \mu f_2 + \nu f_3]_{\alpha, \beta, \gamma}, \ldots, a_{n-k} = \frac{1}{k!}$$

$$\times [\lambda f_1 + \mu f_2 + \nu f_3]^{(k)}_{\alpha, \beta, \gamma}, \ldots, a_0 = \frac{1}{n!} [\lambda f_1 + \mu f_2 + \nu f_3]^{(n)}_{\alpha, \beta, \gamma},$$

f_1, f_2, and f_3 are the partial derivatives of $f(x, y, z)$ with respect to x, y, and z respectively, and the symbolic powers of the trinomial are obtained from the ordinary, non-symbolic powers by replacing products of the form $f_1{}^p f_2{}^q f_3{}^r$ by the partial derivatives $\dfrac{\partial^{p+q+r} f}{\partial x^p\, \partial y^q\, \partial z^r}$.

COROLLARY 1. **A straight line has at most n points in common with a surface which is the locus of an equation of degree n in x, y, and z, unless the entire line lie in the surface.**

For the equation (1), being of degree n in s, has at most n real roots unless it be satisfied by every value of s; and to every real root of (1) there corresponds one point that is common to the line and the surface.

DEFINITION IV. **A *surface of order n* is the locus of an equation of degree n in x, y, and z.**

Remark. It follows from Corollary 1, in combination with this definition, that if a surface is met by no line in more than n points, its order is at most equal to n. The number of different points which a line actually has in common with a surface depends on the number of distinct, finite, real roots of equation (1); in every numerical case this number can be determined by the methods developed in the theory of algebraic equations. If two or more roots of the equation are equal (let their common value be s^*) we say that this number of points of intersection of the line and the surface coincide at the point $(\alpha + \lambda s^*, \beta + \mu s^*, \gamma + \nu s^*)$. Conversely, the algebraic meaning to be attached to the statement that "a line $x = \alpha + \lambda s$, $y = \beta + \mu s$, $z = \gamma + \nu s$ meets a surface $f(x, y, z) = 0$ in two or more coincident points" is that the equation (1) has a double, or a multiple root. To complex roots of the equation (1) correspond complex points of intersection of the line and the surface, that is, points whose coördinates are complex numbers. To values of λ, μ, and ν for which $a_0 = 0$, corresponds an infinite root of the equation.

COROLLARY 2. **The line whose equations are $x = \alpha + \lambda s$, $y = \beta + \mu s$, $z = \gamma + \nu s$ lies entirely on the surface whose equation is $f(x, y, z) = 0$ if and only if**

$$f(\alpha, \beta, \gamma) = 0, \quad \lambda f_1(\alpha, \beta, \gamma) + \mu f_2(\alpha, \beta, \gamma) + \nu f_3(\alpha, \beta, \gamma) = 0,$$
$$[\lambda f_1 + \mu f_2 + \nu f_3]^{(2)}_{\alpha, \beta, \gamma} = 0, \ldots, [\lambda f_1 + \mu f_2 + \nu f_3]^{(n)}_{\alpha, \beta, \gamma} = 0.$$

This corollary is an immediate consequence of Theorem 5.

77. Tangent Lines and Tangent Planes. Normals.

DEFINITION V. **A line is *tangent to a surface at a point* $A(\alpha, \beta, \gamma)$ if at least two of the points of intersection of the line with the surface coincide at the point A.**

This definition leads, by way of Theorem 5 (Section 76, page 153), to the following theorem:

THEOREM 6. **If the point $A(\alpha, \beta, \gamma)$ lies on the surface $f(x, y, z) = 0$, the line $x = \alpha + \lambda s$, $y = \beta + \mu s$, $z = \gamma + \nu s$ will be tangent to this surface at A if and only if the direction cosines λ, μ, ν satisfy the condition:**

(1) $$\lambda f_1(\alpha, \beta, \gamma) + \mu f_2(\alpha, \beta, \gamma) + \nu f_3(\alpha, \beta, \gamma) = 0.$$

Proof. Since the point A lies on the surface, the coefficient a_n in equation (1) of Section 76 vanishes; hence $s = 0$ is one root of this equation and to this value of s corresponds the point A on the line. The line is tangent to the surface at A if and only if at least one other of its intersections with the surface falls at A, that is, if the equation has 0 as a root of multiplicity 2 at least (see Remark in Section 76). But this is equivalent to the requirement that $a_{n-1} = [\lambda f_1 + \mu f_2 + \nu f_3]_{\alpha, \beta, \gamma} = 0$; the theorem is therefore proved.

We ask next for the locus of all points P such that the lines PA which join P to a fixed point A on the surface shall be tangent to the surface at A. If the coördinates of P are x, y, z, the direction cosines of AP are proportional to $x - \alpha$, $y - \beta$, and $z - \gamma$ (see Corollary 1 of Theorem 6, Chapter III, Section 33, page 56). The equation of the required locus is therefore obtained if $x - \alpha$, $y - \beta$, $z - \gamma$ are substituted for λ, μ, ν in condition (1) of Theorem 6. But the resulting equation is linear in x, y, and z; its locus is therefore a plane. The discussion shows that the plane defined in the next definition actually exists, if the function $f(x, y, z)$ possesses partial derivatives at the point $A(\alpha, \beta, \gamma)$,* not all of which are zero.

DEFINITION VI. **A *plane* is *tangent to a surface at a point* A if every line in the plane which passes through A is tangent to the surface at A, and if every line which is tangent to the surface at A lies in this plane. A line through a point A on a surface is *normal to the surface* at A if it is perpendicular to a plane tangent to the surface at A.**

We can now state the following results:

THEOREM 7. **The plane tangent to the surface $f(x, y, z) = 0$ at the point $A(\alpha, \beta, \gamma)$ on the surface is the locus of the equation**

$$(x - \alpha)f_1(\alpha, \beta, \gamma) + (y - \beta)f_2(\alpha, \beta, \gamma) + (z - \gamma)f_3(\alpha, \beta, \gamma) = 0.$$

* The possibility of non-existence of the partial derivatives of a function, which is suggested here, need not disturb the reader at the present stage. For every function considered in this book, we shall assume that partial derivatives exist at every point. It must however be recognized that this involves an assumption and that the reader is accepting its justifiability on faith. For further treatment of this question the reader is referred to books on the Theory of Functions of a Real Variable.

COROLLARY. **The equations of the normal to the surface $f(x, y, z) = 0$ at the point on the surface $A(\alpha, \beta, \gamma)$ are**

$$\frac{x-\alpha}{f_1(\alpha, \beta, \gamma)} = \frac{y-\beta}{f_2(\alpha, \beta, \gamma)} = \frac{z-\gamma}{f_3(\alpha, \beta, \gamma)}$$

and the parametric equations of the normal are

$$x = \alpha + f_1(\alpha, \beta, \gamma) \cdot t, \quad y = \beta + f_2(\alpha, \beta, \gamma) \cdot t, \quad z = \gamma + f_3(\alpha, \beta, \gamma) \cdot t.$$

78. Exercises.

1. Determine the points in which the surface $2x^2 - y^2 + 4z^2 + 3yz + 6zx + 4xy - 2x + y - 4z + 1 - 0$ is met by the lines:

(a) $x = 9 - 4t, \quad y = 1 - t, \quad z = -7 + 3t;$
(b) $x = -3 + 2t, \quad y = -13 + 6t, \quad z = -3 + t;$
(c) $x = -3 - 2t, \quad y = 9 + 5t, \quad z = 7 + 4t.$

2. Determine the points which are common to the surface $y^2 - 10xz - 8yz - 12x - 17y + 16z + 30 = 0$ and the lines:

(a) $x = 3 - 2t, \quad y = -1 + 2t, \quad z = 2 - t;$
(b) $x = -1 + 3t, \quad y = 3 - t, \quad z = 2t;$
(c) $x = 1 + 3t, \quad y = -6t, \quad z = -3 - 2t;$
(d) $x = 1 - 2t, \quad y = 1 + 2t, \quad z = 2 - t.$

3. Proceed similarly with the surface $3x^2 - 4z^2 + 3yz + 2xy + 4x - 2y + 4z + 2 = 0$ and the lines:

(a) $x = 1 + 3t, \quad y = 5 + 4t, \quad z = 6 + 6t;$
(b) $x = -2 + t, \quad y = 1 - t, \quad z = 2t;$
(c) $x = 10t, \quad y = 2t, \quad z = 17t.$

4. Expand by use of Taylor's theorem:

(a) $2(x + h)^3 - 4(x + h)(y + k) + 6(z + l)^2 - 3(x + h) + 5;$
(b) $3(x - h)^2 + 5(x - h)(z - l) - 4(y - k)^2 + 2(x - h) + 3(z - l) - 7.$

5. Determine the condition which the direction cosines of a line through the point $A(1, -1, -1)$ on the surface $3x^2 - 4yz + 2z^2 - 4x + 2y + 5 = 0$ must satisfy in order that the line may be tangent to the surface at A. Write the equations for each of two mutually perpendicular tangent lines to the surface at A.

6. Set up the conditions which the direction cosines of a line through the point $A(-1, 4, 3)$ on the surface $3x^2 - 2y^2 + 3z^2 - 24x - 4y - 12z + 30 = 0$ must satisfy in order that the entire line may lie on the surface; determine two lines through A which lie on the surface.

7. Determine the tangent plane and the normal to the surface $2x^2 - y^2 + z^2 - 3zx + 4xy + 3x - 2y - 4 = 0$ at each of the following points on the surface:

$$A(1, 1, 1), \quad B(0, 0, -2), \quad C(0, -2, 2).$$

8. Write the equation of the tangent plane and the equations of the normal to the surface:

$$x^3 + y^3 + z^3 = 1, \text{ at the point } A(-1, 1, 1).$$

9. Determine the condition which the direction cosines of a line through the origin must satisfy in order to lie entirely upon the conical surface $x^2 - 4y^2 + 4z^2 - 2yz + 2zx + 4xy = 0$ (that is, to be a generator of this surface); determine the generators of this surface which lie in the coördinate planes.

10. Prove that the origin lies on every plane that is tangent to a conical surface whose vertex is at the origin. Extend this proposition so as to show that the vertex of a conical surface lies on every plane that is tangent to the surface.

79. The Shape of a Curve in Space. To determine the locus of a plane curve in space, the methods developed in Chapter V are sufficient. For, by means of them we can transform the frame of reference in such a manner as to make the plane in which the curve lies one of the coördinate planes in the new frame. When this has been accomplished the curve can be studied by the methods of Plane Analytical Geometry.

For curves, whose points do not all lie in one plane (such curves are usually called **twisted curves,** in French ***courbes gauches***) a representation by means of a plane drawing can be obtained by the use of Descriptive Geometry or of Perspective Drawing. Some idea of the shape of the curve can also be obtained from drawings which show the projections of the curve on the three coördinate planes.

Certain general properties of the curve can be detected by the principles stated in Theorem 4 (Section 70, page 137). There should be no difficulty in seeing, for example, that the curve determined by the equations $\frac{x^2}{4} + \frac{y^2}{6} + \frac{z^2}{9} = 1$ and $x^2 + y^2 = 1$, which is the intersection of an ellipsoid and a circular cylinder, is symmetric with respect to the three coördinate planes, the three coördinate axes and the origin. The Z-projection of this curve is given by the equations $x^2 + y^2 = 1, z = 0$; the Y-projection by the equations $\frac{x^2}{10} + \frac{2z^2}{15} = 1,\ y = 0$; the X-projection by the equations $\frac{4z^2}{27} - \frac{y^2}{9} = 1$.

For a more detailed study of the properties of curves and sur-

faces of general character, the reader is referred to treatises on Differential Geometry. In the next Chapters we shall take up the study of the loci of equations of the second degree in x, y, and z.

CHAPTER VII

QUADRIC SURFACES, GENERAL PROPERTIES

Surfaces which are loci of equations of the second degree in x, y, and z are surfaces of the second order, see Definition IV, Chapter VI, Section 76, page 154; they are usually called **quadric surfaces** or **conicoids.** A number of such surfaces have already been discussed, as to their shape, in Chapter VI (see Sections 68 and 72). In proceeding to a more detailed study of these surfaces, we shall apply to them the results obtained in Sections 76 and 77.

80. The Quadric Surface and the Line. We shall write the general equation of the second degree in the form

$$Q(x, y, z) \equiv a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2\,a_{23}yz + 2\,a_{31}zx + 2\,a_{12}xy \\ + 2\,a_{14}x + 2\,a_{24}y + 2\,a_{34}z + a_{44} = 0.$$

We shall find it convenient to use the symbol $Q(x, y, z)$, or simply Q, throughout as an abbreviation for this general form of the function of the second degree in x, y, and z; we shall also use Q to designate the quadric surface which is the locus of the equation $Q(x, y, z) = 0$. The notations a_{23} and a_{32}, a_{31} and a_{13}, a_{12} and a_{21}, a_{14} and a_{41}, a_{24} and a_{42}, a_{34} and a_{43} will be used interchangeably. The partial derivatives of Q will be denoted by the subscript notation, as already agreed upon in Section 75 (see footnote on page 151). Moreover we shall use $q(x, y, z)$, or q, to designate the part of Q which is homogeneous of the second degree; and q with subscripts will be used for the partial derivatives of q. Thus we have

$$q(x, y, z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2\,a_{23}yz + 2\,a_{31}zx + 2\,a_{12}xy;$$
$$Q_1 = 2(a_{11}x + a_{12}y + a_{13}z + a_{14});$$
$$Q_2 = 2(a_{21}x + a_{22}y + a_{23}z + a_{24});$$
$$Q_3 = 2(a_{31}x + a_{32}y + a_{33}z + a_{34});$$
$$q_1 = 2(a_{11}x + a_{12}y + a_{13}z), \quad q_2 = 2(a_{21}x + a_{22}y + a_{23}z). \quad q_3 = 2(a_{31}x + a_{32}y + a_{33}z);$$
$$q_{11} = Q_{11} = 2\,a_{11}, \quad q_{22} = Q_{22} = 2\,a_{22}, \quad q_{33} = Q_{33} = 2\,a_{33}, \quad q_{23} = Q_{23} = 2\,a_{23}, \quad q_{31} = Q_{31} = 2\,a_{31}, \quad q_{12} = Q_{12} = 2\,a_{12}.$$

Furthermore we shall find it useful to use Q_4 as an abbreviation for the linear expression $2(a_{41}x + a_{42}y + a_{43}z + a_{44})$, and correspondingly q_4 as an abbreviation for the linear homogeneous part of Q_4, that is, for $2(a_{41}x + a_{42}y + a_{43}z)$; and the derivatives of these expressions will be denoted by the use of a double subscript on Q or on q. We observe that the partial derivatives of Q and of q of order higher than the second are identically zero.

From Theorem 5, Chapter VI (Section 76, page 153), we derive therefore the following results.

THEOREM 1. **The parameter values of the points in which the line $x = \alpha + \lambda s$, $y = \beta + \mu s$, $z = \gamma + \nu s$ meets the quadric surface $Q(x, y, z) = 0$ are the roots of the equation $L_0 s^2 + 2L_1 s + L_2 = 0$, where**

$$L_2 = Q(\alpha, \beta, \gamma), \quad L_1 = \frac{\lambda Q_1(\alpha, \beta, \gamma) + \mu Q_2(\alpha, \beta, \gamma) + \nu Q_3(\alpha, \beta, \gamma)}{2}$$
$$= \lambda(a_{11}\alpha + a_{12}\beta + a_{13}\gamma + a_{14}) + \mu(a_{21}\alpha + a_{22}\beta + a_{23}\gamma + a_{24}) + \nu(a_{31}\alpha + a_{32}\beta + a_{33}\gamma + a_{34}),$$

and $L_0 = q(\lambda, \mu, \nu) = a_{11}\lambda^2 + a_{22}\mu^2 + a_{33}\nu^2 + 2a_{23}\mu\nu + 2a_{31}\nu\lambda + 2a_{12}\lambda\mu.$

COROLLARY 1. **The line $x = \alpha + \lambda s$, $y = \beta + \mu s$, $z = \gamma + \nu s$ will (a) meet the quadric surface Q in two distinct real points, if and only if $L_1^2 - L_0L_2 > 0$; (b) be tangent to Q if and only if $L_1^2 = L_0L_2$; (c) not have any real points in common with the surface if and only if $L_1^2 - L_0L_2 < 0$.**

COROLLARY 2. **The line $x = \alpha + \lambda s$, $y = \beta + \mu s$, $z = \gamma + \nu s$ will lie entirely on the quadric surface Q if and only if $L_0 = L_1 = L_2 = 0$.**

Remark. Values of λ, μ, ν for which $L_0 = q(\lambda, \mu, \nu) = 0$ give rise to at least one infinite root of the equation. Such values correspond to lines which meet the surface in one or more infinitely distant points. A line for which $L_1 = L_0 = 0$, but $L_2 \neq 0$ meets the curve in two infinitely distant points. We lay down now the following definition.

DEFINITION I. **A direction determined by values of λ, μ, ν which satisfy the equation $q(\lambda, \mu, \nu) = 0$ is called an *asymptotic direction* of the quadric surface Q. A line which meets a quadric surface in two infinitely distant points is called an *asymptote* of the surface.**

We can now state a further corollary.

COROLLARY 3. **The necessary and sufficient conditions that the line $x = \alpha + \lambda s$, $y = \beta + \mu s$, $z = \gamma + \nu s$ be an asymptote of the quadric surface Q are that $q(\lambda, \mu, \nu) = 0$ and that $\lambda Q_1(\alpha, \beta, \gamma) + \mu Q_2(\alpha, \beta, \gamma) + \nu Q_3(\alpha, \beta, \gamma) = 0$.**

81. Tangent Line; Tangent Plane; Normal; Polar Plane. If the discussion of Section 77 be applied to the quadric surface Q, the following results will appear without difficulty.

THEOREM 2. **If the point $A(\alpha, \beta, \gamma)$ lies on the quadric Q, the line $x = \alpha + \lambda s$, $y = \beta + \mu s$, $z = \gamma + \nu s$ will be tangent to the surface at A if and only if its direction cosines λ, μ, ν satisfy the equation $\lambda Q_1(\alpha, \beta, \gamma) + \mu Q_2(\alpha, \beta, \gamma) + \nu Q_3(\alpha, \beta, \gamma) = 0$.**

Remark. This result can also be obtained from Corollary 1 of Theorem 1.

THEOREM 3. **The equation of the plane tangent to the quadric surface Q at the point $A(\alpha, \beta, \gamma)$ on the surface is $(x - \alpha)\, Q_1(\alpha, \beta, \gamma) + (y - \beta)\, Q_2(\alpha, \beta, \gamma) + (z - \gamma)\, Q_3(\alpha, \beta, \gamma) = 0$.**

COROLLARY. **The equations of the normal to the quadric surface Q at the point $A(\alpha, \beta, \gamma)$ are $x = \alpha + Q_1(\alpha, \beta, \gamma) \cdot t$, $y = \beta + Q_2(\alpha, \beta, \gamma) \cdot t$, $z = \gamma + Q_3(\alpha, \beta, \gamma) \cdot t$.**

The equation of the tangent plane to the quadric surface, given in Theorem 3, can be put in a form which is very convenient for application in numerical cases. We observe first that $Q(\alpha, \beta, \gamma) = q(\alpha, \beta, \gamma) + 2\,a_{14}\alpha + 2\,a_{24}\beta + 2\,a_{34}\gamma + a_{44}$ and that therefore

$$Q_1 = q_1 + 2\,a_{14}, \quad Q_2 = q_2 + 2\,a_{24}, \quad Q_3 = q_3 + 2\,a_{34}.$$

And it follows from the notation introduced in the first paragraph of Section 80 that $Q_4 = q_4 + 2\,a_{44}$. Furthermore, since q is a homogeneous function of the second degree, it follows* that $\alpha q_1(\alpha, \beta, \gamma) + \beta q_2(\alpha, \beta, \gamma) + \gamma q_3(\alpha, \beta, \gamma) = 2 \cdot q(\alpha, \beta, \gamma)$. If we add $2(2\,a_{14}\alpha + 2\,a_{24}\beta + 2\,a_{34}\gamma + a_{44}) = 2\,q_4(\alpha, \beta, \gamma) + 2\,a_{44} = 2\,Q_4(\alpha, \beta, \gamma) - 2\,a_{44}$ to both sides of the last equation, we find that

$$\alpha Q_1(\alpha, \beta, \gamma) + \beta Q_2(\alpha, \beta, \gamma) + \gamma Q_3(\alpha, \beta, \gamma) + Q_4(\alpha, \beta, \gamma) = 2\,Q(\alpha, \beta, \gamma).$$

The left-hand side of the equation of the tangent plane, as given in Theorem 3, may now be transformed as follows, remembering

* We are here making use of the following theorem, known as **Euler's theorem on homogeneous functions:** If $F(x, y, z)$ is a homogeneous function of degree n in x, y, and z, then $xF_1 + yF_2 + zF_3 = nF$. The proof of this theorem may be made as follows: The homogeneity of $F(x, y, z)$ tells us that $F(kx, ky, kz) = k^n \cdot F(x, y, z)$ for every k. Differentiation with respect to k gives $xF_1(kx, ky, kz) + yF_2(kx, ky, kz) + zF_3(kx, ky, kz) = nk^{n-1} \cdot F(x, y, z)$, from which the formula of the theorem is obtained if we substitute 1 for k.

that $Q(\alpha, \beta, \gamma) = 0$:

$$\begin{aligned}(x - \alpha)Q_1(\alpha, \beta, \gamma) &+ (y - \beta)Q_2(\alpha, \beta, \gamma) + (z - \gamma)Q_3(\alpha, \beta, \gamma)\\ &= xQ_1 + yQ_2 + zQ_3 + Q_4 = 2[x(a_{11}\alpha + a_{12}\beta + a_{13}\gamma + a_{14})\\ &+ y(a_{21}\alpha + a_{22}\beta + a_{23}\gamma + a_{24}) + z(a_{31}\alpha + a_{32}\beta + a_{33}\gamma\\ &+ a_{34}) + a_{41}\alpha + a_{42}\beta + a_{43}\gamma + a_{44}].\end{aligned}$$

If we remember the convention concerning the coefficients a_{ij}, which was made in the first paragraph of Section 80, we have the following result:

Theorem 4. **The tangent plane to the quadric surface Q at the point $A(\alpha, \beta, \gamma)$ on the surface, has the equation**

$$\begin{aligned}a_{11}\alpha x + a_{22}\beta y + a_{33}\gamma z + a_{23}(\beta z + \gamma y) + a_{31}(\gamma x + \alpha z) + a_{12}(\alpha y + \beta x)\\ + a_{14}(x + \alpha) + a_{24}(y + \beta) + a_{34}(z + \gamma) + a_{44} = 0.\end{aligned}$$

Remark. The reader will observe that this form of the equation of the tangent plane is obtainable from the equation of the quadric surface by a process which consists in distributing the coefficients of the equation among the variables x, y, z and the constants α, β, γ in equal shares; the technical name of this process is "polarization."

Corollary 1. **If α, β, γ are selected entirely arbitrarily, we have the relation**

$$\alpha Q_1(\alpha, \beta, \gamma) + \beta Q_2(\alpha, \beta, \gamma) + \gamma Q_3(\alpha, \beta, \gamma) + Q_4(\alpha, \beta, \gamma) = 2\,Q(\alpha, \beta, \gamma).$$

The equation in Theorem 4 represents a plane whether $A(\alpha, \beta, \gamma)$ is on the quadric surface Q or not. For a general position of the point A, this plane is called the polar plane of A with respect to the surface.

Definition II. **The plane which is represented by the equation $a_{11}\alpha x + a_{22}\beta y + a_{33}\gamma z + a_{23}(\beta z + \gamma y) + a_{31}(\gamma x + \alpha z) + a_{12}(\alpha y + \beta x) + a_{14}(x + \alpha) + a_{24}(y + \beta) + a_{34}(z + \gamma) + a_{44} = 0$ is the *polar plane of the point* $A(\alpha, \beta, \gamma)$ with respect to the quadric surface Q; the point is called the *pole of the plane* with respect to the surface.**

This definition enables us to state a further corollary of Theorem 4, namely,

Corollary 2. **The tangent plane to the quadric surface Q at the point $A(\alpha, \beta, \gamma)$ on the surface coincides with the polar plane of A with respect to the quadric surface.**

Thus we have obtained a geometrical interpretation of the polar plane of a point on the surface Q with respect to this surface. We

proceed next to inquire as to the geometrical significance of the polar plane with respect to the surface Q of a point which is not on this surface.

We observe first that, in view of the proof of Theorem 4, of Corollary 1 of this theorem, and of Definition II, the equation of the polar plane can be written in the form

$$xQ_1(\alpha,\ \beta,\ \gamma) + yQ_2(\alpha,\ \beta\ \gamma) + zQ_3(\alpha,\ \beta,\ \gamma) + Q_4(\alpha,\ \beta,\ \gamma) = 0$$

or in the equivalent form

$$(1)\quad (x - \alpha)Q_1(\alpha,\ \beta,\ \gamma) + (y - \beta)Q_2(\alpha,\ \beta,\ \gamma) + (z - \gamma)\ Q_3(\alpha,\ \beta,\ \gamma) + 2\,Q(\alpha,\ \beta,\ \gamma) = 0.$$

Let us now consider an arbitrary line l through the point $A(\alpha,\ \beta,\ \gamma)$:

$$x = \alpha + \lambda s, \quad y = \beta + \mu s, \quad z = \gamma + \nu s;$$

let the points where the line l meets the quadric surface be A_1 and A_2; and let its point of intersection with the polar plane (1) of A with respect to the surface be A' (see Fig. 31). It follows then from the geometrical meaning of the parameter s in the equations of the line l (see Corollary 2 of Theorem 10, Chapter III, Section 34, page 60) that the directed segments AA_1 and AA_2 are equal to the roots of the equation $L_0s^2 + 2L_1s + L_2 = 0$, established in Theorem 1; and that the directed segment AA' is equal to the root of the equation

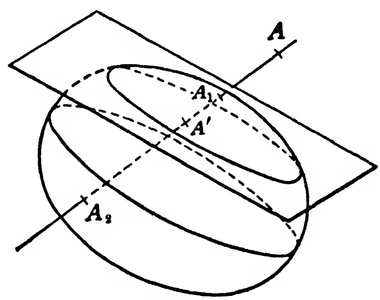

FIG. 31

$$\lambda sQ_1(\alpha, \beta, \gamma) + \mu sQ_2(\alpha, \beta, \gamma) + \nu sQ_3(\alpha, \beta, \gamma) + 2\,Q(\alpha, \beta, \gamma) = 0,$$

obtained by eliminating x, y, and z between the equations of the line l and the equation of the polar plane (1). Now we have to recall the meaning of the coefficients L_0, L_1 and L_2 stated in Theorem 1, and also the relations between the coefficients and the roots of an algebraic equation (the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to $-\frac{b}{a}$, their product is equal

to $\frac{c}{a}\Big)$. With the aid of these tools, we find the following result:

$$AA' = -\frac{2Q}{\lambda Q_1 + \mu Q_2 + \nu Q_3}\bigg|_{\alpha,\beta,\gamma} = -\frac{2L_2}{2L_1} = 2\frac{L_2/L_0}{-2L_1/L_0}$$
$$= \frac{2 \cdot AA_1 \cdot AA_2}{AA_1 + AA_2}.$$

From this relation we derive two consequences:

(*a*) If we divide both sides by two and take the reciprocals of both sides of the equation, we find that $\frac{2}{AA'} = \frac{1}{AA_1} + \frac{1}{AA_2}$, or that

$$\frac{1}{AA_1} - \frac{1}{AA'} = \frac{1}{AA'} - \frac{1}{AA_2}.$$

This informs us that, independently of the direction of the line l, the reciprocals of the segments AA_1, AA', and AA_2 form an arithmetical progression, that is, the segments AA_1, AA', and AA_2 form a harmonic progression, so that AA' is the harmonic mean between AA_1 and AA_2.

(*b*) If we transform the relation by writing $AA' = AA_1 + A_1A'$, clearing of fractions and carrying out the indicated operations, we find that

$$(AA_1)^2 + AA_1 \cdot A_1A' + AA_2 \cdot A_1A' = AA_1 \cdot AA_2;$$

that is,

$$AA_1(A_2A + AA_1 + A_1A') + AA_2 \cdot A_1A' = 0,$$

or $AA_1 \cdot A_2A' + AA_2 \cdot A_1A' = 0$,

or finally $\frac{A_1A}{AA_2} = -\frac{A_1A'}{A'A_2}$.

This equation expresses the fact that the points A and A' divide the segment A_1A_2 in ratios which are equal numerically, but opposite in sign. This is what is meant by the statement that A' and A are harmonic conjugates with respect to the points A_1 and A_2, according to the following definition:

Definition III. **If A, B, C, and D are collinear points and so situated that the ratio of the segments CA and AD, in which A divides the segment CD, is equal numerically but opposite in sign to the ratio of the segments CB and BD in which B divides the segment CD, then A and B are called *harmonic conjugates* with respect to C and D.**

It should be clear from the preceding discussion that if A and B are harmonic conjugates with respect to C and D, then the segment AB is a harmonic mean between the segments AC and AD, and conversely. Furthermore, if B is the harmonic conjugate of A with respect to the intersections of the line AB with the quadric surface $Q(x, y, z) = 0$, then $AB = -\left.\dfrac{2\,Q}{\lambda Q_1 + \mu Q_2 + \nu Q_3}\right|_{\alpha,\beta,\gamma}$ in which λ, μ, ν are the direction cosines of the line AB. It follows from this that the coördinates of B are

$$x_B = \alpha - \left.\frac{2\,\lambda Q}{\lambda Q_1 + \mu Q_2 + \nu Q_3}\right|_{\alpha,\beta,\gamma},$$
$$y_B = \beta - \left.\frac{2\,\mu Q}{\lambda Q_1 + \mu Q_2 + \nu Q_3}\right|_{\alpha,\beta,\gamma},$$
$$z_B = \gamma - \left.\frac{2\,\nu Q}{\lambda Q_1 + \mu Q_2 + \nu Q_3}\right|_{\alpha,\beta,\gamma}.$$

Consequently

$$(x_B - \alpha)Q_1(\alpha, \beta, \gamma) + (y_B - \beta)Q_2(\alpha, \beta, \gamma) + (z_B - \gamma)Q_3(\alpha, \beta, \gamma)$$
$$= -\left.\frac{2\,Q(\lambda Q_1 + \mu Q_2 + \nu Q_3)}{\lambda Q_1 + \mu Q_2 + \nu Q_3}\right|_{\alpha,\beta,\gamma} = -2\,Q(\alpha, \beta, \gamma),$$

from which we conclude that the coördinates of B satisfy the equation (1) of the polar plane. We can summarize our discussions by the following theorem.

THEOREM 5. **The polar plane of a point A with respect to a quadric surface is the locus of the harmonic conjugates of A with respect to the intersections of the surface with the lines through A.**

82. Polar Plane and Pole. Tangent Cone. Preliminary to a discussion of some further properties of the polar plane we raise the question whether it is possible for a plane to have more than one pole with respect to a quadric surface, that is, whether it is possible for two different points $A(\alpha, \beta, \gamma)$ and $A'(\alpha', \beta', \gamma')$ to have the same polar plane with respect to the surface. In view of Definition II this amounts to the question whether the equations

$$(a_{11}\alpha + a_{12}\beta + a_{13}\gamma + a_{14})x + (a_{12}\alpha + a_{22}\beta + a_{23}\gamma + a_{24})\,y + (a_{13}\alpha + a_{23}\beta + a_{33}\gamma + a_{34})z + (a_{14}\alpha + a_{24}\beta + a_{34}\gamma + a_{44}) = 0,$$
$$(a_{11}\alpha' + a_{12}\beta' + a_{13}\gamma' + a_{14})x + (a_{12}\alpha' + a_{22}\beta' + a_{23}\gamma' + a_{24})\,y + (a_{13}\alpha' + a_{23}\beta' + a_{33}\gamma' + a_{34})z + (a_{14}\alpha' + a_{24}\beta' + a_{34}\gamma' + a_{44}) = 0$$

can represent the same plane. This will be the case if and only if the coefficients of x, y, and z and the constant terms in the two equations differ by a factor of proportionality, k (see the footnote on page 78), that is, if and only if the numbers $k\alpha - \alpha'$, $k\beta - \beta'$, $k\gamma - \gamma'$ and $k - 1$ satisfy the following system of linear homogeneous equations:

$$\begin{aligned}
a_{11}(k\alpha - \alpha') + a_{12}(k\beta - \beta') + a_{13}(k\gamma - \gamma') + a_{14}(k - 1) &= 0,\\
a_{12}(k\alpha - \alpha') + a_{22}(k\beta - \beta') + a_{23}(k\gamma - \gamma') + a_{24}(k - 1) &= 0,\\
a_{13}(k\alpha - \alpha') + a_{23}(k\beta - \beta') + a_{33}(k\gamma - \gamma') + a_{34}(k - 1) &= 0,\\
a_{14}(k\alpha - \alpha') + a_{24}(k\beta - \beta') + a_{34}(k\gamma - \gamma') + a_{44}(k - 1) &= 0.
\end{aligned}$$

This system of equations will or will not possess a nontrivial solution, according as its coefficient determinant has a value that is equal to or different from zero (see Theorem 2, Chapter II, page 38). In the latter case, the only solution is the trivial one, so that we find $k\alpha - \alpha' = k\beta - \beta' = k\gamma - \gamma' = k - 1 = 0$; from this we find $k = 1$, $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, so that no plane can have more than one pole. The discussion leads to the following definitions and conclusions:

Definition IV. **The determinant of the symmetric square matrix $\|a_{ij}\|$, $i, j = 1, 2, 3, 4$, $a_{ij} = a_{ji}$, is called the *discriminant* of the quadric surface Q; we shall use Δ to designate the value of this determinant.**

Definition V. **A quadric surface whose discriminant vanishes is called a *singular quadric surface*.** (Compare Definitions II and III on page 43.)

Theorem 6. **No plane has more than one pole with respect to a non-singular quadric surface.**

Remark 1. The homogeneous equation of the second degree in x, y, and z represents a conical surface of the second order; it is called a **quadric cone**. The quadric cone is a singular quadric surface, for in its equation $a_{14} = a_{24} = a_{34} = a_{44} = 0$; consequently the value of the discriminant is zero.

Remark 2. While Theorem 6 asserts that no plane has more than one pole with respect to a non-singular quadric surface, it does not say that there exists a pole for *every* plane in space. The question depends on whether the system of equations

$$\begin{aligned}
a_{11}\alpha + a_{12}\beta + a_{13}\gamma + a_{14} &= ka,\\
a_{12}\alpha + a_{22}\beta + a_{23}\gamma + a_{24} &= kb,\\
a_{13}\alpha + a_{23}\beta + a_{33}\gamma + a_{34} &= kc,\\
a_{14}\alpha + a_{24}\beta + a_{34}\gamma + a_{44} &= kd,
\end{aligned}$$

does or does not have a solution for α, β, γ, and k for every set of values of a, b, c, and d. Since for a non-singular quadric the augmented matrix of this system is always of rank 4, it follows from Theorems 1 and 8 of Chapter II (see Sections 21 and 27, pages 36 and 44) that a plane will have a single pole or none with respect to a non-singular quadric surface, according as the value of the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a \\ a_{12} & a_{22} & a_{23} & b \\ a_{13} & a_{23} & a_{33} & c \\ a_{14} & a_{23} & a_{34} & d \end{vmatrix}$ is different from or equal to zero.

We derive now a number of further consequences from the definition of the polar plane.

THEOREM 7. **If $A(\alpha, \beta, \gamma)$ lies on the polar plane of $A'(\alpha', \beta', \gamma')$ with respect to a quadric surface, then A' lies on the polar plane of A with respect to the same surface.**

The proof of this theorem is left to the reader.

THEOREM 8. **If $A(\alpha, \beta, \gamma)$ lies on its own polar plane with respect to a quadric surface, then A lies on the surface; and conversely.**

The first part of this statement becomes evident when we substitute the coördinates of A in the equation of the polar plane of A; the second part follows from Corollary 2 of Theorem 4 (Section 81, page 162).

Suppose now that from a point A not on the quadric Q tangents be drawn to the surface; let the points of contact of these tangents be A', B', etc. Then, since A lies in the planes tangent to the surface at A', B', etc., A lies in the polar planes of A', B', etc., with respect to the surface. It follows therefore from Theorem 7, that the polar plane A passes through A', B', etc. Conversely, if A' is a point in which the polar plane of A meets the surface, then the tangent plane to the surface at A' passes through A, that is, the line AA' is tangent to the surface. We have therefore the following theorem:

THEOREM 9. **The points of contact of a quadric surface Q with the tangents drawn from a point A not on the surface, are the points in which the surface is met by the polar plane of A with respect to Q.**

On the basis of this result we introduce the following definition:

DEFINITION VI. **The *tangent cone* from a point A to a non-singular quadric surface which does not contain A is the cone whose vertex is at A and whose directrix is the curve common to the surface and to the polar plane of A.**

Remark. This tangent cone is a quadric cone; for the curve in which a plane meets a quadric surface can not be of higher order than the second; and since A does not lie on the surface, it can not lie on its own polar plane.

To obtain the equation of the tangent cone, we shall use two methods.

(*a*) In the first method, we translate the frame of reference to A as a new origin, set up the equation of the cone in the new system and then return to the original frame. Since the transformation of coördinates has been discussed in Chapter V, we shall suppose that the translation has already been carried out and we shall ask therefore for the tangent cone from the origin O to a quadric surface Q which does not pass through the origin.

It follows from Theorem 9 that the cone passes through the points common to the surface and the polar plane of the origin; consequently its equation can be written in the form

$$(1) \quad k_1Q(x, y, z) + k_2(a_{14}x + a_{24}y + a_{34}z + a_{44}) = 0$$

(see Remark 4, Section 49, page 92). The multipliers k_1 and k_2 have not been restricted in any manner as yet; since we are looking however for a quadric cone, they have to be selected in such a way that equation (1) shall be a homogeneous equation of the second degree in x, y, and z. Therefore we put $k_1 = k$ and $k_2 = lx + my + nz + p$ and we write down that the constant term and the coefficients of the first degree terms in equation (1) must vanish; this leads to the following conditions:

$$(2\,k + p)a_{14} + la_{44} = 0, \quad (2\,k + p)a_{24} + ma_{44} = 0,$$
$$(2\,k + p)a_{34} + na_{44} = 0, \quad ka_{44} + pa_{44} = 0.$$

Since the origin does not lie on the surface Q, $a_{44} \neq 0$; hence the last equation gives $p = -k$. Substituting this value for p in the other equations, we find $la_{44} = -ka_{14}$, $ma_{44} = -ka_{24}$ and $na_{44} = -ka_{34}$. We are therefore free to choose an arbitrary non-zero value for k, as could be expected from the fact that equation (1) could have been divided through by k_1 without essentially changing anything. Taking $k = a_{44}$, we obtain $l = -a_{14}$, $m = -a_{24}$, $n = -a_{34}$, $p = -a_{44}$. Consequently the tangent cone from the origin to the non-singular quadric Q is represented by the homogeneous equation $a_{44}Q(x, y, z) - (a_{14}x + a_{24}y + a_{34}z + a_{44})^2 = 0$.

(*b*) In the second method we suppose that $P(x, y, z)$ is an arbitrary point on the tangent cone. Then the line AP is tangent to the surface and its direction cosines must therefore satisfy the condition of Corollary 1 of Theorem 1 (Section 80, page 160), namely:

$$\tfrac{1}{4}[\lambda Q_1(\alpha, \beta, \gamma) + \mu Q_2(\alpha, \beta, \gamma) + \nu Q_3(\alpha, \beta, \gamma)]^2 = Q(\alpha, \beta, \gamma) \times (a_{11}\lambda^2 + a_{22}\mu^2 + a_{33}\nu^2 + 2\,a_{23}\mu\nu + 2\,a_{31}\nu\lambda + 2\,a_{12}\lambda\mu).$$

But for the line AP, we have $\lambda : \mu : \nu = x - \alpha : y - \beta : z - \gamma$; and since the condition which we have just written down is homogeneous of the second degree in λ, μ, and ν, we may omit the factor of proportionality. We obtain therefore for the required tangent cone the following equation homogeneous of the second degree in $x - \alpha$, $y - \beta$, and $z - \gamma$:

$$\tfrac{1}{4}[(x - \alpha)Q_1(\alpha, \beta, \gamma) + (y - \beta)Q_2(\alpha, \beta, \gamma) + (z - \gamma)Q_3(\alpha, \beta, \gamma)]^2 = Q(\alpha, \beta, \gamma) \cdot [a_{11}(x - \alpha)^2 + a_{22}(y - \beta)^2 + a_{33}(z - \gamma)^2 + 2\,a_{23}(y - \beta)(z - \gamma) + 2\,a_{31}(z - \gamma)(x - \alpha) + 2\,a_{12}(x - \alpha)(y - \beta)].$$

Remark. For $\alpha = \beta = \gamma = 0$, this last equation should be equivalent to the equation obtained by the first method.

83. Exercises.

1. Determine the equation of the tangent plane and the equations of the normal for the surface $4\,x^2 - 6\,xy + 5\,y^2 + 4\,yz - 3\,z^2 + 2\,zx - 4\,x + 3\,y + 2\,z + 4 = 0$ at the point $A(1, -1, 2)$.

2. Set up the condition which the direction cosines of a line through $P(2, -1, 1)$ must satisfy in order to be tangent to the surface $3\,x^2 - 2\,y^2 + 5\,zx - 4\,y + 6\,z - 3 = 0$.

3. Set up the condition on λ, μ, ν under which the line $x = -1 + \lambda s$, $y = 2 + \mu s$, $z = -2 + \nu s$ will lie entirely on the surface $4\,x^2 - 6\,y^2 + 8\,z^2 = 12$.

4. Determine the polar plane with respect to the ellipsoid $3\,x^2 + 2\,y^2 + 4\,z^2 = 20$ of the points $A(-2, 2, 1)$, $B(5, 5, 0)$, $C(0, 4, -3)$, $D(0, 0, 0)$.

5. Find the pole with respect to the surface $3\,x^2 - 2\,xy + y^2 + 4\,yz - 6\,x + 2\,y + 7 = 0$ of the plane (*a*) $x + y + z - 3 = 0$; (*b*) $2\,x - y + 2\,z + 3 = 0$.

6. Derive the equation of the tangent cone to the surface $4\,x^2 + 3\,y^2 = 12\,z$ from the points $A(0, 0, -6)$, $B(-4, 5, 3)$, $C(0, 0, 4)$.

7. Determine the equation of the tangent cone from an arbitrary point $P(\alpha, \beta, \gamma)$ to (*a*) the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; (*b*) the hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$; (*c*) the elliptic paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2\,pz$.

8. Show that the equation for the tangent cone obtained by the second method of the last part of Section 82 reduces to the result found by the first method if we put $\alpha = \beta = \gamma = 0$.

9. Determine the asymptotic directions of the hyperboloid of one sheet $\frac{x^2}{4} - \frac{y^2}{6} + \frac{z^2}{8} = 1$, which lie in the plane $3x - 2y = 0$.

10. Show that the asymptotes of an hyperbola are also asymptotes of the hyperboloid of revolution of one sheet which is obtained when the hyperbola is revolved about the conjugate axis.

11. Show that the ellipsoid of Exercise 7 does not have any real asymptotic directions.

12. Determine whether the hyperboloid of two sheets $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, the elliptic paraboloid of Exercise 7 or the hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2pz$ have real asymptotic directions.

84. Ruled Quadric Surfaces. We have already met a few examples of surfaces which contain every point of a line (see, for example, Exercise 3 in Section 83). We proceed now to a systematic study of the question which quadric surfaces have straight lines on them. We saw in Corollary 2 of Theorem 1 (Section 80, page 160) that the line $x = \alpha + \lambda s$, $y = \beta + \mu s$, $z = \gamma + \nu s$ will lie entirely on the quadric surface Q if and only if $L_0 = q(\lambda, \mu, \nu) = a_{11}\lambda^2 + a_{22}\mu^2 + a_{33}\nu^2 + 2a_{23}\mu\nu + 2a_{31}\nu\lambda + 2a_{12}\lambda\mu = 0$,

$$L_1 = \tfrac{1}{2} \cdot [\lambda Q_1(\alpha, \beta, \gamma) + \mu Q_2(\alpha, \beta, \gamma) + \nu Q_3(\alpha, \beta, \gamma)] = 0$$
$$\text{and } L_2 = Q(\alpha, \beta, \gamma) = 0.$$

Suppose now that we have a point $A(\alpha, \beta, \gamma)$ on the surface, so that the condition $L_2 = 0$ is satisfied. To determine lines through A which lie entirely on the surface, λ, μ, and ν have to be so selected that $L_1 = L_0 = 0$. These equations are homogeneous in λ, μ, and ν and of degree 1 and 2 respectively; if we solve them for two of the variables, say λ and μ, in terms of the third variable ν, we shall in general be led to a quadratic equation, giving rise to two values for the ratios $\lambda : \mu : \nu$, that is, to two lines on the surface through A. These two values may be real and distinct, coincident or imaginary. We want to learn under which conditions these different situations will arise.

We begin by proving the following auxiliary theorem.

THEOREM 10. **A non-singular quadric Q contains no point at which $Q_1 = Q_2 = Q_3 = 0$.**

Proof. It follows from Corollary 1 of Theorem 4 (Section 81, page 162) that at a point at which $Q = Q_1 = Q_2 = Q_3 = 0$, we must also have $Q_4 = 0$. Hence there would be at least one set of three numbers α, β, γ which satisfy the four linear non-homogeneous equations $a_{i_1}\alpha + a_{i_2}\beta + a_{i_3}\gamma + a_{i_4} = 0$, $i = 1, 2, 3, 4$. If the rank of the coefficient matrix of this system of equations is 3, a solution exists if and only if the rank of the augmented matrix is also 3 (see Theorem 8, Chapter II, Section 27, page 44). But the determinant of this augmented matrix is the discriminant of the quadric surface (see Definition IV, Section 82, page 166); hence if the rank of the augmented matrix is 3, the surface is singular. On the other hand, if the rank of the coefficient matrix of the system of linear equations is less than three, then the cofactors of the elements in the last column of the discriminant are all equal to zero and the value of the discriminant is therefore zero. Thus we have seen that if a quadric surface contains a point at which Q_1, Q_2 and Q_3 all vanish, then it is singular.

We shall have frequent occasion to refer to a point on a quadric at which these conditions hold and we introduce therefore a name for it.

Definition VII. **A *vertex* of a quadric surface Q is a point at which $Q = Q_1 = Q_2 = Q_3 = 0$.**

In the terminology of this definition, we can then state the following corollary:

Corollary. **A non-singular quadric surface has no vertex; a singular quadric surface may have one or more vertices.**

We proceed now with the problem of determining lines through a point $A(\alpha, \beta, \gamma)$ on a quadric which shall lie entirely on the surface; and we shall divide our discussion of this question in two parts:

Case I. The point $A(\alpha, \beta, \gamma)$ is not a vertex.

In this case at least one of the partial derivatives of Q is different from zero at A; let us suppose that $Q_1(\alpha, \beta, \gamma) \neq 0$. We can then solve the equation $L_1 = 0$ for λ in terms of μ and ν and substitute the result in the equation $L_0 = 0$. This will lead us to the following quadratic equation in μ and ν:*

$$(1)\quad \begin{vmatrix} a_{11} & a_{12} & Q_1 \\ a_{12} & a_{22} & Q_2 \\ Q_1 & Q_2 & 0 \end{vmatrix} \mu^2 + 2 \begin{vmatrix} a_{11} & a_{12} & Q_1 \\ a_{13} & a_{23} & Q_3 \\ Q_1 & Q_2 & 0 \end{vmatrix} \mu\nu + \begin{vmatrix} a_{11} & a_{13} & Q_1 \\ a_{13} & a_{33} & Q_3 \\ Q_1 & Q_3 & 0 \end{vmatrix} \nu^2 = 0,$$

* In order not to interrupt our main argument too much at this point, we relegate the proof of this statement to the Appendix, I (see page 296).

in which the partial derivatives Q_1, Q_2, and Q_3 have the arguments α, β, and γ; it will be understood that this is the case throughout our further argument, unless the contrary is definitely specified.

We observe now that the coefficients of μ^2, $2\mu\nu$, and ν^2 in this equation are equal respectively to the minors of the elements a_{33}, a_{23}, and a_{22} in the determinant

$$(2) \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} & Q_1 \\ a_{12} & a_{22} & a_{23} & Q_2 \\ a_{13} & a_{23} & a_{33} & Q_3 \\ Q_1 & Q_2 & Q_3 & 0 \end{vmatrix}.$$

If the value of this determinant is denoted by $A_3(Q)$ and the cofactors of its elements a_{ij} by $A_{ij}(Q)$, it follows from Theorem 18, Chapter I (see Section 16, page 29), that the discriminant of equation (1) is equal to $A_{23}{}^2(Q) - A_{22}(Q) \times A_{33}(Q)$

$$= - \begin{vmatrix} A_{22}(Q) & A_{23}(Q) \\ A_{23}(Q) & A_{33}(Q) \end{vmatrix} = - A_3(Q) \cdot \begin{vmatrix} a_{11} & Q_1 \\ Q_1 & 0 \end{vmatrix} = Q_1{}^2 \cdot A_3(Q).$$

Consequently the roots of the quadratic equation (1) will be real and unequal, real and equal or complex according as the value of the determinant (2) is positive, zero or negative. We will show now that this determinant can be reduced to a simpler form. If we add to the last column the products of the 1st, 2nd, and 3rd columns by -2α, -2β, and -2γ respectively, its elements will become $2a_{14}, 2a_{24}, 2a_{34}$ and $-2\alpha Q_1 - 2\beta Q_2 - 2\gamma Q_3$. Next, we add to the last row the products of the 1st, 2nd and 3rd rows by -2α, -2β, and -2γ respectively; this transforms the elements of the last row into $2a_{14}$, $2a_{24}$, $2a_{34}$ and $-2\alpha Q_1 - 2\beta Q_2 - 2\gamma Q_3 - 4\alpha a_{14} - 4\beta a_{24} - 4\gamma a_{34}$. This last element, in the lower right-hand corner, is equal to $-2(\alpha Q_1 + \beta Q_2 + \gamma Q_3 + Q_4) + 4a_{44} = 4(a_{44} - Q)$, by use of Corollary 1 of Theorem 4 (Section 81, page 162). Therefore the determinant (2) has been reduced to

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & 2a_{14} \\ a_{12} & a_{22} & a_{23} & 2a_{24} \\ a_{13} & a_{23} & a_{33} & 2a_{34} \\ 2a_{14} & 2a_{24} & 2a_{34} & 4(a_{44} - Q) \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & 2a_{14} \\ a_{12} & a_{22} & a_{23} & 2a_{24} \\ a_{13} & a_{23} & a_{33} & 2a_{34} \\ 2a_{14} & 2a_{24} & 2a_{34} & 4a_{44} \end{vmatrix} +$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{12} & a_{22} & a_{23} & 0 \\ a_{13} & a_{23} & a_{33} & 0 \\ 2a_{14} & 2a_{24} & 2a_{34} & -4Q \end{vmatrix}.$$

Therefore, if, in agreement with our general notation, we designate by A_{44} the cofactor of the element a_{44} in the discriminant Δ of the quadric surface, we obtain the interesting formula

$$(3)\quad A_3(Q) = 4\,\Delta - 4\,A_{44}\cdot Q(\alpha,\,\beta,\,\gamma).$$

In the particular case which we are having under consideration, $Q(\alpha, \beta, \gamma) = 0$, since the point $A(\alpha, \beta, \gamma)$ lies on the surface and therefore $A_3(Q) = 4\,\Delta$. We state this preliminary result of our discussion in the following theorem and corollary.

THEOREM 11. **If the point $A(\alpha, \beta, \gamma)$ lies on the quadric surface Q, then the value of the determinant $A_3(Q)$ is independent of the position of A on the surface and equal to four times the value of the discriminant Δ of the surface.**

COROLLARY. **The matrices of the determinant $A_3(Q)$ and of the discriminant Δ have equal rank, if the point $A(\alpha, \beta, \gamma)$ lies on the quadric Q.**

Proof. This Corollary follows from Theorem 14, Chapter I, in view of the fact that if $Q(\alpha, \beta, \gamma) = 0$, the matrix of the determinant $A_3(Q)$ is transformed into that of the discriminant Δ by means of elementary transformations (see Definition XIV, Chapter I, Section 10, page 18).

The further discussion of our problem depends on the rank of the matrix of the discriminant Δ; henceforth we shall denote this matrix by the symbol $\mathbf{a}_4$ and its rank by r_4. We consider now the following possibilities:

(*a*) $r_4 = 4$, that is, $\Delta \neq 0$.

It follows from our discussion that in this case the quadratic equation (1) will have two distinct roots, which are real if $\Delta > 0$ and complex if $\Delta < 0$; to each root of the equation (1) there corresponds a set of direction cosines of a line through A which will lie entirely on the surface. We can conclude therefore that through every point on a quadric surface for which $\Delta > 0$, there pass two different real lines which lie entirely on the surface; and that through no point on a surface for which $\Delta < 0$ there are lines which lie on the surface.

(*b*) $r_4 = 3$.

As a result of the Corollary of Theorem 11, the rank of the matrix of the determinant (2) will also be equal to 3 in that case.

We can show now that the coefficients of equation (1) can not all vanish, by showing that if they did, then every three-rowed minor of the determinant (2) would vanish.* Consequently in this case the equation (1) has two coincident roots and through every point of the surface, which is not a vertex, there will pass two coincident lines which lie entirely on the surface.

(*c*) $r_4 = 2$ or 1.

It follows now from the Corollary of Theorem 11, that all the coefficients of the equation (1) vanish. Consequently every line through the point A whose direction cosines satisfy the condition $L_1 = 0$ lies entirely on the surface. But this carries with it that every line through A which lies in the plane $(x - \alpha)Q_1(\alpha, \beta, \gamma) + (y - \beta)Q_2(\alpha, \beta, \gamma) + (z - \gamma)Q_3(\alpha, \beta, \gamma) = 0$ must lie on the surface. We conclude that the surface contains every point of this plane; in virtue of Corollary 1 of Theorem 4 (Section 81, page 162) and because $Q(\alpha, \beta, \gamma) = 0$, the equation of this plane may also be written in the form $xQ_1(\alpha, \beta, \gamma) + yQ_2 + zQ_3 + Q_4 = 0$.

Case II. The point $A(\alpha, \beta, \gamma)$ is a vertex of the surface.

In this case, which can arise only on a singular quadric (see Corollary of Theorem 10, page 171), the equation $L_1 = 0$ is satisfied by every set of direction cosines. And we shall show that the condition $L_0 = q(\lambda, \mu, \nu) = 0$ is satisfied by the direction cosines of any line which joins $A(\alpha, \beta, \gamma)$ to another point $A'(\alpha', \beta', \gamma')$ on the surface and by no others. For, in virtue of Taylor's theorem (see Section 75, page 151) we have

$$\begin{aligned} Q(\alpha', \beta', \gamma') &= Q(\alpha + [\alpha' - \alpha], \beta + [\beta' - \beta], \gamma + [\gamma' - \gamma]) \\ &= Q(\alpha, \beta, \gamma) + [(\alpha' - \alpha)Q_1(\alpha, \beta, \gamma) + (\beta' - \beta)Q_2(\alpha, \beta, \gamma) \\ &\quad + (\gamma' - \gamma)Q_3(\alpha, \beta, \gamma)] + q(\alpha' - \alpha, \beta' - \beta, \gamma' - \gamma). \end{aligned}$$

Therefore, if $A(\alpha, \beta, \gamma)$ is a vertex of the surface and if $A'(\alpha', \beta', \gamma')$ is an arbitrary second point on the surface, then $q(\alpha' - \alpha, \beta' - \beta, \gamma' - \gamma) = 0$. But since the direction cosines of the line AA' are proportional to $\alpha' - \alpha$, $\beta' - \beta$, and $\gamma' - \gamma$, and since q is a homogeneous function, it follows that $q(\lambda, \mu, \nu) = 0$. And it should be an easy matter to show that this will not be the case for the direction cosines of a line which connects the vertex A with any other point in space.

* The proof is given in the Appendix, II (see page 296).

The results of the discussion of our problem may now be summarized as follows.

THEOREM 12. **Through every real point $A(\alpha, \beta, \gamma)$ on a non-singular quadric surface with positive discriminant, there pass two and only two lines which lie entirely on the surface; through no real point on a non-singular quadric surface with negative discriminant is there any line which lies entirely on the surface. Through every point on a singular quadric for which the rank of the matrix of the discriminant is 3, and which is not a vertex of this surface, there pass two coincident lines which lie entirely on the surface, and no others. Through every point on a singular quadric for which the rank of the discriminant matrix is less than 3, and which is not a vertex of this surface, there passes a plane which belongs entirely to the surface. The lines joining a vertex of a singular quadric to any other point on the surface lie entirely on the surface.**

COROLLARY 1. **A singular quadric surface which possesses a vertex is a conical surface; it is a quadric cone.**

For, from the last part of Theorem 12, it follows that the surface may be generated by a line through a vertex which moves so as to pass through the points of the surface cut out by any plane which does not pass through the vertex.

Remark. A vertex of a singular quadric is also a vertex of the quadric cone which it represents.

COROLLARY 2. **If the rank of the discriminant matrix of a quadric surface is less than 3, the locus of the equation consists of two planes; it is a degenerate quadric.** (See Definition V, Chapter IV, Section 46, page 83.)

Proof. For it follows from Theorem 12 that in this case there is at least one plane all of whose points lie on the surface. Let the left-hand side of the equation of this plane be E; and let $Q = E \cdot E_1 + R$, where R is a function of y and z alone. Obviously if R does not vanish identically we can determine particular values of y and z for which $R \neq 0$; and it will also be possible in general to associate with these values of y and z a value of x such that these values of x, y, and z cause E to vanish. But for these values, we will have $Q \neq 0$; and therefore, we would have a point on the plane $E = 0$ which does not lie on the surface. This contradicts our hypothesis. Consequently, R must vanish identically, and $Q = E\, E_1$. It is now easy to see that E_1 is also a linear function and therefore we conclude by use of Theorem 10, Chapter IV

(Section 46, page 83) that the locus of $Q = 0$ consists of two planes.

85. The Centers and Vertices of Quadric Surfaces. Among the particular quadric surfaces with which we have already become familiar are the sphere, the elliptic cylinder and the circular cone. The center of a sphere is usually defined as the point from which all the points on the sphere are equally distant; for an elliptic cylinder, and even for a circular cylinder such a point does not exist. If we take for the center of the sphere however the property that it bisects every chord which passes through it, we observe that every elliptic cylinder has points which possess the same property, namely, the points on its axis. The axis of such a cylinder could then be called a line of centers. But even on this definition of a center, the cone, the elliptic paraboloid and other quadrics do not possess any centers. We undertake therefore in the present section the inquiry as to the conditions under which a quadric has a center; and we shall seek to develop convenient methods for the location of centers in the cases in which they exist. Our work will be based on the following definition:

Definition VIII. **A *center* of a quadric surface is a point which bisects every chord drawn through it;* a proper center is a center which does not lie on the surface, an improper center of a surface lies on the surface.**

It follows from this definition that, if $A(\alpha, \beta, \gamma)$ is a center of the quadric surface Q, then the two roots of the equation

$$L_0 s^2 + 2L_1 s + L_2 = 0$$

which was established in Section 80, must be equal numerically but opposite in sign for all admissible values of λ, μ, and ν (see the Remark, following Theorem 7, Chapter III, Section 33, page 56); hence the sum of these roots must equal 0.

If $L_0 \neq 0$, that is, if the line through $A(\alpha, \beta, \gamma)$ does not have an asymptotic direction (see Definition I, Section 80, page 160), the sum of the roots is equal to $-\dfrac{2L_1}{L_0}$; it will be equal to zero therefore

* A chord of a surface is a line which joins two of its points; it follows from Corollary 1 of Theorem 1, Section 80, page 160, that a chord of a quadric surface does not have any other points in common with the surface besides the two points which it joins, unless it lie entirely on the surface.

if and only if $L_1 = 0$. And if $L_0 = 0$, so that one of the roots is infinite, the condition requires that the other root be also infinite, which leads again to the condition $L_1 = 0$. Conversely, if $L_1 = 0$, the two roots of equation (1) are equal numerically but opposite in sign. Therefore the necessary and sufficient condition that $A(\alpha, \beta, \gamma)$ be a center is that $L_1 = 0$ for all admissible values of λ, μ, ν. In particular we must have $L_1 \equiv \lambda Q_1(\alpha, \beta, \gamma) + \mu Q_2(\alpha, \beta, \gamma) + \nu Q_3(\alpha, \beta, \gamma) = 0$ for the sets of values 1, 0, 0; 0, 1, 0 and 0, 0, 1 of λ, μ, ν; these special sets lead to the conditions $Q_1(\alpha, \beta, \gamma) = 0$, $Q_2(\alpha, \beta, \gamma) = 0$, $Q_3(\alpha, \beta, \gamma) = 0$. Moreover it is easily seen that if these conditions are fulfilled, then L_1 will vanish for every admissible set of values of λ, μ, ν. We have therefore obtained the following theorem:

THEOREM 13. **The necessary and sufficient conditions that a point shall be a center of the quadric surface Q is that its coördinates satisfy the three linear equations $Q_1(x, y, z) = 0$, $Q_2(x, y, z) = 0$, $Q_3(x, y, z) = 0$. If and only if the coördinates satisfy moreover the condition $Q(x, y, z) \neq 0$, the point is a proper center.**

COROLLARY. **An improper center of a quadric surface is a vertex of the surface, and conversely.**

For the further discussion of our problem we observe in the first place that, in view of Corollary 1 of Theorem 4, the condition $Q \neq 0$ of Theorem 13 may be replaced by the condition $Q_4 \neq 0$. Consequently a proper center is a point common to the three planes

$$\frac{Q_1}{2} = a_{11}x + a_{12}y + a_{13}z + a_{14} = 0, \qquad \frac{Q_2}{2} = a_{12}x + a_{22}y + a_{23}z + a_{24} = 0, \quad \frac{Q_3}{2} = a_{13}x + a_{23}y + a_{33}z + a_{34} = 0,$$

but *not* on the plane

$$\frac{Q_4}{2} = a_{14}x + a_{24}y + a_{34}z + a_{44} = 0;$$

and a vertex is a point common to the four planes.

We shall denote the coefficient matrix of the first three equations by $\mathbf{a}_3$ and its rank by r_3; the value of the determinant of $\mathbf{a}_3$ has already been designated by A_{44} (see equation (3), Section 84, page 173). The augmented matrix of the first three equations is

$||a_{ij}||$, $i = 1, 2, 3$; $j = 1, 2, 3, 4$; we shall denote it by $\mathbf{b}$. The coefficient matrix of the equations of the set of four planes is $||a_{ij}||$, $i = 1, 2, 3, 4$; $j = 1, 2, 3$; we shall denote it by $\mathbf{b}'$; and the augmented matrix of this set of four equations is the discriminant matrix of the quadric, which we have already designated by $\mathbf{a}_4$ (see page 173, proof of Corollary of Theorem 11) and whose determinant is denoted by Δ.

We fall back now on Theorems 20, 22, 23, 24 of Chapter IV and their Corollaries (Sections 51 and 54, pages 95, 101, and 102); application of these theorems shows that if there is to be any center the matrices $\mathbf{a}_3$ and $\mathbf{b}$ must have the same rank, and if there are to be any vertices, the matrices $\mathbf{b}'$ and $\mathbf{a}_4$ must have the same rank.

Now it should be clear: (1) that the ranks of the matrices $\mathbf{b}$ and $\mathbf{b}'$ are equal, since either of these matrices is obtained from the other if we write the columns as rows and vice versa; (2) that the rank of $\mathbf{a}_3$ can not exceed that of $\mathbf{b}$, which in turn can not exceed the rank of $\mathbf{a}_4$; (3) that the rank of $\mathbf{a}_4$ can not exceed the rank of $\mathbf{b}$ by more than 1; and (4) that the rank of $\mathbf{b}$ can not exceed the rank of $\mathbf{a}_3$ by more than 1. Moreover, (5) if the ranks of $\mathbf{b}$ and $\mathbf{a}_4$ are equal, then the ranks of $\mathbf{b}$ and $\mathbf{a}_3$ are equal.

An algebraic proof of this last statement may be somewhat lengthy. It can be deduced very readily however from the theorems of Chapter IV referred to above. For if the ranks of $\mathbf{b}$ and $\mathbf{a}_4$ are equal, the four planes have at least one point in common; consequently the first three planes have at least one point in common and therefore the c.m. and the a.m. of the first three equations have the same rank, i.e., the ranks of $\mathbf{a}_3$ and $\mathbf{b}$ are equal.

We conclude from (3) and (4) that r_4 and r_3 can differ by 2 at most. If $r_4 - r_3 = 2$, then the rank of $\mathbf{b}$ is different from either. If $r_4 - r_3 = 1$, it follows from (5) that the rank of $\mathbf{b}$ is equal to r_3. And, if $r_4 = r_3$, the rank of $\mathbf{b}$ is of course equal to the same number. It should be clear that the existence of proper centers and vertices depends on the difference $r_4 - r_3$. If we draw on the further content of the Theorems of Chapter IV, which were cited above, we obtain the following result:

THEOREM 14. **If the ranks of the matrices $\mathbf{a}_4$ and $\mathbf{a}_3$ are equal the quadric surface has a unique vertex, a line of vertices or a plane of vertices, according as this common rank is 3, 2 or 1. If the ranks of these matrices differ by 1, the quadric surface will have a single**

proper center, a line of proper centers or a plane of proper centers, according as the lower of these ranks is 3, 2 or 1. If the ranks of these matrices differ by 2, the quadric surface has no center at all.

Remark 1. The content of this theorem may conveniently be put in the following tabular form:

r_3	r_4	The quadric surface has
3	4	a single proper center
3	3	a single vertex
2	3	a line of proper centers
2	2	a line of vertices
1	2	a plane of proper centers
1	1	a plane of vertices
$r_4 - r_3 > 1$		no center

Remark 2. The reader should convince himself that the cases indicated in this table include all possible cases for the ranks of the matrices $\mathbf{a}_3$ and $\mathbf{a}_4$ and that therefore the conditions of Theorem 14 are sufficient as well as necessary.

Remark 3. A quadric surface with a single proper center is called a **central quadric.** A quadric surface with a single vertex is called a **proper quadric cone.**

Remark 4. A non-singular quadric surface is either a central quadric or else a surface without any center.

We record moreover the following corollaries.

Corollary 1. **The rank of the matrix $\mathbf{a}_4$ can not exceed the rank of the matrix $\mathbf{a}_3$ by more than 2.**

Corollary 2. **The necessary and sufficient condition that a quadric surface be a conical surface is that the ranks of the matrices $\mathbf{a}_4$ and $\mathbf{a}_3$ be equal.**

We are able furthermore to complete in an essential way the result contained in Corollary 2 of Theorem 12 (Section 84, page 175), as follows: If a quadric surface has a plane of vertices, it consists of this plane, counted doubly. For, if it contained a point A outside this plane it would have to contain every line which connects a point of the plane with A (compare the last sentence in Theorem 12, Section 84, page 175). This is obviously impossible; therefore the surface can not contain any point outside the plane of vertices. And it should be a simple matter to show that

this plane must be counted doubly. And if a quadric surface has a line of vertices, it must consist of two planes through this line. For, if A is any point of the surface outside the line l on which the vertices lie, then the plane determined by l and A must be entirely contained in the surface; and the argument used in the proof of Corollary 2 of Theorem 12 (page 175) shows that then the surface consists of two planes. If these two planes were coincident planes the equation of the surface would be $Q = (ax + by + cz + d)^2 = 0$, from which we could conclude that $r_3 = r_4 = 1$, and therefore the surface would have a plane of vertices. We can therefore state the following result:

Corollary 3. **A singular quadric surface is a proper quadric cone if and only if $r_3 = r_4 = 3$, a pair of intersecting planes if and only if $r_3 = r_4 = 2$, a pair of coincident planes if and only if $r_3 = r_4 = 1$.**

After the existence of centers or vertices has been established, their position can be determined by solving the equations $Q_1 = 0$, $Q_2 = 0$, $Q_3 = 0$. In the case of a central quadric, these equations have an unique solution which is given by Cramer's rule (see Theorem 1, Chapter II, Section 21, page 37). The solution may be written in the following form:

$$x : y : z : 1 = -\begin{vmatrix} a_{14} & a_{12} & a_{13} \\ a_{24} & a_{22} & a_{23} \\ a_{34} & a_{32} & a_{33} \end{vmatrix} : -\begin{vmatrix} a_{11} & a_{14} & a_{13} \\ a_{12} & a_{24} & a_{23} \\ a_{13} & a_{34} & a_{33} \end{vmatrix}$$
$$: -\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{12} & a_{22} & a_{24} \\ a_{13} & a_{23} & a_{34} \end{vmatrix} : \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}.$$

It should be easy to see that the terms on the right are equal to the cofactors of the elements in the last row of the discriminant Δ. If these cofactors are designated in the usual manner, we have $x : y : z : 1 = A_{14} : A_{24} : A_{34} : A_{44}$.

Corollary 4. **The coördinates of the center of a central quadric surface are equal to $\frac{A_{14}}{A_{44}}, \frac{A_{24}}{A_{44}}, \frac{A_{34}}{A_{44}}$.**

Examples.

1. The coördinates of the possible centers of the surface $5x^2 + 5y^2 + 8z^2 + 8yz + 8zx - 2xy + 12x - 12y + 6 = 0$ must satisfy the equations

$$\frac{Q_1}{2} = 5x - y + 4z + 6 = 0, \quad \frac{Q_2}{2} = -x + 5y + 4z - 6 = 0,$$
$$\frac{Q_3}{2} = 4x + 4y + 8z = 0.$$

Moreover, $\frac{Q_4}{2} = 6x - 6y + 6.$

The rank of the matrix $\mathbf{a}_3 = \begin{Vmatrix} 5 & -1 & 4 \\ -1 & 5 & 4 \\ 4 & 4 & 8 \end{Vmatrix}$ is readily found to be 2; for the 3rd row is equal to the sum of the first two rows, whereas the two-rowed minor in the upper left-hand corner does not vanish. And the rank of the matrix $\mathbf{a}_4 = \begin{Vmatrix} 5 & -1 & 4 & 6 \\ -1 & 5 & 4 & -6 \\ 4 & 4 & 8 & 0 \\ 6 & -6 & 0 & 6 \end{Vmatrix}$ is found to be 3; for the 3rd row is equal to the sum of the 1st and 2nd rows, whereas the matrix contains several non-vanishing three-rowed minors. We conclude therefore that the surface has a line of centers in the line of intersection of the two planes $\frac{Q_1}{2} = 5x - y + 4z + 6 = 0$ and $\frac{Q_2}{2} = -x + 5y + 4z - 6 = 0$. The parametric equations of this line are found to be $x = 3 + t$, $y = 5 + t$, $z = -4 - t$.

2. To determine the possible centers of the surface $2x^2 - 3y^2 + 4yz - 5zx + 4x - 3y + 5 = 0$, we set up the equations $Q_1 = 4x - 5z + 4 = 0$, $Q_2 = -6y + 4z - 3 = 0$, $Q_3 = -5x + 4y = 0$, and $Q_4 = 4x - 3y + 10 = 0$. The determinant of the matrix $\mathbf{a}_3$ has the value:

$$\frac{1}{8} \times \begin{vmatrix} 4 & 0 & -5 \\ 0 & -6 & 4 \\ -5 & 4 & 0 \end{vmatrix} = \frac{43}{4};$$

and the discriminant

$$\Delta = \frac{1}{16} \times \begin{vmatrix} 4 & 0 & -5 & 4 \\ 0 & -6 & 4 & -3 \\ -5 & 4 & 0 & 0 \\ 4 & -3 & 0 & 10 \end{vmatrix} = \frac{861}{16}.$$

Therefore $r_3 = 3$ and $r_4 = 4$; consequently the surface has a single proper center; its coördinates are $\frac{2}{43}$, $\frac{5}{86}$, $\frac{36}{43}$.

3. For the surface $2x^2 + 20y^2 + 18z^2 - 12yz + 12xy + 22x + 6y - 2z - 2 = 0$, the matrices $\mathbf{a}_3$ and $\mathbf{a}_4$ are

$$\begin{Vmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & -6 & 18 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} 2 & 6 & 0 & 11 \\ 6 & 20 & -6 & 3 \\ 0 & -6 & 18 & -1 \\ 11 & 3 & -1 & -2 \end{Vmatrix}$$

respectively. We find that $r_3 = 2$ and $r_4 = 4$; we conclude that the surface has neither a proper center nor a vertex.

86. Exercises.

1. Show that through every point of the surface $4x^2 - 6y^2 = 12z$ there pass two real distinct lines which lie entirely on the surface.

2. Prove that there are no real lines on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

3. Show that the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, the hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, and the hyperboloid of two sheets $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ are central quadric surfaces.

4. Show that through every point of the hyperboloid of one sheet of Exercise 3 there pass two real lines which lie on the surface; and that no such lines exist through any point of the hyperboloid of two sheets of Exercise 3.

5. Show that the locus of the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$ is a proper quadric cone; and prove that every tangent plane of this surface passes through the vertex.

6. Determine the conditions which the direction cosines of a line through the point $A(-1, -1, 1)$ on the surface $x^3 - y^3 + z^3 = 1$ must satisfy in order that the line may lie entirely on the surface.

7. Determine the centers, proper centers or vertices, of each of the following surfaces:

(a) $x^2 + 5y^2 - 2z^2 + 6yz + 8xy - 4x + 6y - 6z + 6 = 0$
(b) $9x^2 + 49y^2 + 4z^2 - 28yz + 12zx - 42xy - 24x + 56y - 16z + 16 = 0$
(c) $3x^2 + 5y^2 + 9z^2 + 2yz + 8zx - 4xy - 6x + 4y - 4z + 3 = 0$
(d) $5x^2 - y^2 - 16z^2 - 20yz + 4zx - 8xy - 6x + 2y - 8z + 2 = 0$
(e) $4x^2 + y^2 + 9z^2 - 6yz + 12zx - 4xy + 6x - 3y + 9z - 4 = 0$
(f) $6x^2 - 2y^2 - 2z^2 + 5yz - zx - 4xy - 10x - 6y + 9z - 4 = 0$
(g) $3x^2 + 3y^2 + 3z^2 - 2yz - 2zx - 2xy + 8x - 4z + 6 = 0$
(h) $2x^2 + 5y^2 + 2z^2 - 6yz + 4zx - 6xy + 2x - 4y + 2z + 2 = 0$.

8. Show that the tangent lines from a point $A(\alpha, \beta, \gamma)$ to the elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ lie on a pair of planes through a line parallel to the Z-axis.

9. Prove that if a quadric surface has a plane of centers the surface consists of a pair of parallel planes.

10. Prove that the elliptic paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2pz$ and the hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2pz$ do not possess a center.

87. The Asymptotic Cone. If $C(\alpha, \beta, \gamma)$ is the center of the central quadric Q, then $Q_1(\alpha, \beta, \gamma) = 0$, $Q_2(\alpha, \beta, \gamma) = 0$, $Q_3(\alpha, \beta, \gamma) = 0$ and $Q(\alpha, \beta, \gamma) \neq 0$. It follows that the equation of the

tangent cone from C to the surface reduces from the form given at the end of Section 82 (page 169) to the simpler form:

$$(1)\quad a_{11}(x-\alpha)^2 + a_{22}(y-\beta)^2 + a_{33}(z-\gamma)^2 + 2\,a_{23}(y-\beta)(z-\gamma) + 2\,a_{31}(z-\gamma)(x-\alpha) + 2\,a_{12}(x-\alpha)(y-\beta) = 0.$$

Since the direction cosines of a generating line on this cone are proportional to $x-\alpha$, $y-\beta$, and $z-\gamma$, where x, y, z are the coördinates of some point on the cone, it follows that the direction cosines λ, μ, ν of such a line satisfy the equation $a_{11}\lambda^2 + a_{22}\mu^2 + a_{33}\nu^2 + 2\,a_{23}\mu\nu + 2\,a_{31}\nu\lambda + 2\,a_{12}\lambda\mu = 0$, that is, the equation $q(\lambda, \mu, \nu) = 0$. Since moreover they evidently satisfy the equation $L_1 = \lambda Q_1(\alpha, \beta, \gamma) + \mu Q_2 + \nu Q_3 = 0$, the generators of this cone are asymptotes of the surface (compare Corollary 3 of Theorem 1, Section 80, page 160).

Definition IX. **A cone of which every generator is an asymptote of a surface is called an *asymptotic cone* of the surface.**

We can therefore say that the center of a central quadric surface is the vertex of an asymptotic cone of the surface. The same argument shows that a proper center of any non-degenerate quadric surface is the vertex of an asymptotic cone. And we raise the question whether any other points, besides proper centers, can be vertices of such cones of non-degenerate quadrics. If $A(\alpha, \beta, \gamma)$ is such a point, we know from Corollary 3 of Theorem 1 (Section 80, page 160) that the equations $q(\lambda, \mu, \nu) = 0$ and $\lambda Q_1(\lambda, \mu, \nu) + \mu Q_2 + \nu Q_3 = 0$ must have an infinite number of solutions which are admissible values of λ, μ, and ν. Let us suppose now:

(*a*) that $A(\alpha, \beta, \gamma)$ is not a center of the surface. We can then suppose that $Q_1(\alpha, \beta, \gamma) \neq 0$ and proceed as in Section 84. The quadratic equation (1) which was discussed in that section will have more than two roots if and only if the rank r_4 of the discriminant matrix is less than 3 (compare (*b*) on page 173), that is, if the quadric surface is degenerate (see Corollary 2 of Theorem 12, Section 84, page 175). Hence for a non-degenerate non-singular quadric a center is the only point which can be the vertex of an asymptotic cone. And we suppose:

(*b*) that $A(\alpha, \beta, \gamma)$ is a vertex of the surface. In this case Q, Q_1, Q_2, and Q_3 all vanish for $x = \alpha$, $y = \beta$, $z = \gamma$. If we make use once more of Taylor's theorem as in Case II on page 174, we

find that

$$\begin{aligned} Q(x,\ y,\ z) &= Q(\alpha + [x - \alpha],\ \beta + [y - \beta],\ \gamma + [x - \gamma]) \\ &= Q(\alpha, \beta, \gamma) + (x - \alpha)Q_1(\alpha, \beta, \gamma) + (y - \beta)Q_2(\alpha, \beta, \gamma) \\ &+ (z - \gamma)Q_3(\alpha, \beta, \gamma) + q(x - \alpha, y - \beta, z - \gamma), \end{aligned}$$

so that the equation of the surface reduces to the equation $q(x - \alpha, y - \beta, z - \gamma) = 0$, which is the equation of the asymptotic cone.

If we observe furthermore that it follows from the Taylor's expansion written above that the equation of the asymptotic cone of a central quadric can also be written in the form $Q(x, y, z) - Q(\alpha, \beta, \gamma) = 0$, we can put our results in the form of the following theorem.

THEOREM 15. **A non-degenerate quadric surface Q has an asymptotic cone if and only if it has a center. If it has a proper center at $A(\alpha, \beta, \gamma)$ the equation of the asymptotic cone is $Q(x, y, z) - Q(\alpha, \beta, \gamma) = 0$; if it has a vertex the asymptotic cone is identical with the surface itself.**

Obviously there is no further interest in considering the asymptotic cone of a surface which has vertices; therefore there remain for consideration the non-degenerate quadrics which have proper centers, that is, the cases in which r_4 and r_3 are 4 and 3, or 3 and 2 respectively.

CASE 1. $r_4 = 4, r_3 = 3$. In this case there is a single center and therefore a single asymptotic cone, which is a proper quadric cone.

CASE 2. $r_4 = 3, r_3 = 2$. In this case there is a line of centers determined by the equations $Q_1(x, y, z) = 0$, $Q_2(x, y, z) = 0$ and $Q_3(x, y, z) = 0$. The c.m. of these equations is $\mathbf{a}_3$. We shall henceforth denote the cofactors of the elements a_{ij}, $i, j = 1, 2, 3$ of this matrix by α_{ij}, $i, j = 1, 2, 3$. Since $r_3 = 2$, not all of these cofactors vanish. Let us suppose $\alpha_{33} \neq 0$; then the direction cosines of the line of centers are proportional to α_{13}, α_{23}, and α_{33} (see Theorem 17, Chapter IV, Section 47, page 87). Therefore, if α, β, γ is an arbitrary point on the line of centers, the parametric equations of this line may be put in the form $x = \alpha + \alpha_{13}t$, $y = \beta + \alpha_{23}t$, $z = \gamma + \alpha_{33}t$; and the equation of the asymptotic cone which corresponds to an arbitrary center can be put in the form:

$$Q(x, y, z) - Q(\alpha + \alpha_{13}t, \beta + \alpha_{23}t, \gamma + \alpha_{33}t) = 0.$$

If the second term on the left-hand side is expanded by Taylor's theorem, the equation reduces to

$$Q(x, y, z) - Q(\alpha, \beta, \gamma) - t[\alpha_{13}Q_1(\alpha, \beta, \gamma) + \alpha_{23}Q_2(\alpha, \beta, \gamma) + \alpha_{33}Q_3(\alpha, \beta, \gamma)] - t^2\, q(\alpha_{13}, \alpha_{23}, \alpha_{33}) = 0.$$

The coefficient of t is obviously zero; and it is shown in the Appendix* that the coefficient of t^2 also vanishes. Consequently the equation of the asymptotic cone, which corresponds to an arbitrary center, is independent of t; that is, there is only one asymptotic cone. If its equation is written in the form $q(x - \alpha, y - \beta, z - \gamma) = 0$, and we translate the axes to the point (α, β, γ) as origin (see Theorem 2, Chapter V, Section 61, page 115), the equation takes the form $q(x', y', z') = 0$. The discriminant matrix of this equation is $\begin{Vmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{12} & a_{22} & a_{23} & 0 \\ a_{13} & a_{23} & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}$; since in this case $r_3 = 2$, the rank of this discriminant matrix is also 2, and therefore, the asymptotic cone consists of a pair of intersecting planes (compare Corollary 3 of Theorem 14, Section 85, page 180); that is, the asymptotic cone degenerates into a pair of asymptotic planes. We have now obtained the following amplification of the last theorem.

THEOREM 16. **If the ranks r_4 and r_3 are equal to 4 and 3 respectively, the quadric surface Q has a single proper quadric cone as asymptotic cone; this cone may be real or imaginary. If these ranks are equal to 3 and 2 respectively, the surface has a pair of asymptotic planes, which may be real or imaginary. In either case the equation of the asymptotic cone may be written in the form $Q(x, y, z) - Q(\alpha, \beta, \gamma) = 0$, where α, β, γ are the coördinates of a center of the surface.**

88. The Diametral Planes and the Principal Planes of a Quadric Surface. We return once more to the equation in Theorem 1 (Section 80, page 160) and inquire for the locus of points which are midpoints of chords drawn through them in a fixed direction given by the ratios $\lambda : \mu : \nu$. The argument which led us to Theorem 13 in Section 85 shows that if $A(\alpha, \beta, \gamma)$ is a point of this locus, then $\lambda Q_1(\alpha, \beta, \gamma) + \mu Q_2 + \nu Q_3$ must vanish for the specified values of λ, μ, and ν. Since Q_1, Q_2, and Q_3 are linear functions of

* See III, page 297.

α, β, γ, we conclude that the locus is a plane. We obtain therefore immediately the following theorem.

THEOREM 17. **The locus of all points which bisect chords of the quadric surface Q whose direction cosines are λ, μ, ν, is the plane $\lambda Q_1(x, y, z) + \mu Q_2(x, y, z) + \nu Q_3(x, y, z) = 0$.**

DEFINITION X. **The plane which is the locus of the midpoints of a set of parallel chords is called the *diametral plane* of the direction of these chords.**

If we write out in full the expressions for Q_1, Q_2, and Q_3 in the equation of the diametral plane and collect the terms in x, y, and z, the equation of this plane takes the form:

$$(a_{11}\lambda + a_{12}\mu + a_{13}\nu)x + (a_{12}\lambda + a_{22}\mu + a_{23}\nu)y + (a_{13}\lambda + a_{23}\mu + a_{33}\nu)z + a_{14}\lambda + a_{24}\mu + a_{34}\nu = 0,$$

or, using the notation introduced in Section 80,

$$q_1(\lambda, \mu, \nu)x + q_2(\lambda, \mu, \nu)y + q_3(\lambda, \mu, \nu)z + q_4(\lambda, \mu, \nu) = 0.$$

For all values of λ, μ, and ν, this equation represents a plane (for those values for which $q_1(\lambda, \mu, \nu) = q_2(\lambda, \mu, \nu) = q_3(\lambda, \mu, \nu) = 0$, and $q_4(\lambda, \mu, \nu) \neq 0$, this plane is the "plane at infinity," see Remark 2, Section 41, page 73), except for such values as cause q_1, q_2, q_3, and q_4 to vanish simultaneously; but this can not happen for admissible values of λ, μ, and ν unless the rank of the matrix **b** is less than 3 (see Theorem 2, Chapter II, Section 22, page 38; compare also Section 85, page 178). We can therefore state the following corollary.

COROLLARY. **In a quadric surface for which the rank of the matrix b is equal to 3, there exists a diametral plane corresponding to *every direction;* in a quadric surface for which the rank of this matrix is less than 3, this correspondence fails for those directions for which $q_1(\lambda, \mu, \nu) = q_2 = q_3 = q_4 = 0$.**

The correspondence between systems of parallel chords and diametral planes which has been established for quadric surfaces, is an extension to three-space of the correspondence between conjugate diameters in the theory of conic sections; for either of two conjugate diameters is the locus of the midpoints of chords parallel to the other. We recall that in the ellipse and the hyperbola there is one pair of mutually perpendicular conjugate diameters, namely, the axes of these curves. On account of the importance

of these lines in the theory of these curves, we are led to inquire whether there are directions in a quadric surface which are perpendicular to the corresponding diametral planes. To facilitate the discussion, we introduce the following definition.

DEFINITION XI. **A *principal direction* of a quadric surface is a direction which is perpendicular to the corresponding diametral plane; a diametral plane which corresponds to a principal direction is called a *principal plane*.**

According to Corollary 3 of Theorem 7, Chapter IV (Section 44, page 79), the angle between any direction λ, μ, ν and the corresponding diametral plane, when this is a "plane at finite distance," is given by the formula:

$$\sin\theta = \pm\frac{[\lambda q_1(\lambda, \mu, \nu) + \mu q_2 + \nu q_3]}{\sqrt{q_1^2 + q_2^2 + q_3^2}}.$$

Therefore the necessary and sufficient condition that the diametral plane which corresponds to the direction λ, μ, ν shall be a plane at

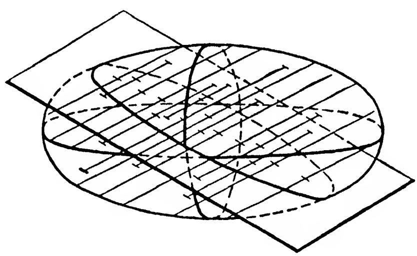

FIG. 32

finite distance and perpendicular to this direction, is that the equation

$$\frac{\lambda q_1(\lambda, \mu, \nu)}{2k} + \frac{\mu q_2(\lambda, \mu, \nu)}{2k} + \frac{\nu q_3(\lambda, \mu, \nu)}{2k} = 1$$

shall be satisfied by admissible values of λ, μ, and ν, such that $k = \pm\frac{\sqrt{q_1^2 + q_2^2 + q_3^2}}{2} \neq 0$ (see Fig. 32).* If we multiply this

* If $k = 0$, we must have $q_1 = q_2 = q_3 = 0$, so that we would be dealing with the plane at infinity if there were a plane at all; and if the diametral plane were the plane at infinity, we would have $q_1 = q_2 = q_3 = 0$ and therefore $k = 0$. Consequently the non-vanishing of k is a necessary and sufficient condition that the diametral plane shall be a plane at finite distance.

equation by 2 and subtract the result from the sum of the equations $\lambda^2 + \mu^2 + \nu^2 = 1$ and $\frac{q_1^2}{4\,k^2} + \frac{q_2^2}{4\,k^2} + \frac{q_3^2}{4\,k^2} = 1$, we obtain the condition

$$\left(\lambda - \frac{q_1}{2\,k}\right)^2 + \left(\mu - \frac{q_2}{2\,k}\right)^2 + \left(\nu - \frac{q_3}{2\,k}\right)^2 = 0,$$

which in turn is equivalent to the three equations

$$q_1(\lambda,\, \mu,\, \nu) = 2\,k\lambda, \quad q_2(\lambda,\, \mu,\, \nu) = 2\,k\mu, \quad q_3(\lambda,\, \mu,\, \nu) = 2\,k\nu;$$

that is, to the equations

$$(1) \quad (a_{11} - k)\lambda + a_{12}\mu + a_{13}\nu = 0, \quad a_{12}\lambda + (a_{22} - k)\mu + a_{23}\nu = 0, \quad a_{13}\lambda + a_{23}\mu + (a_{33} - k)\nu = 0^*.$$

The condition for the existence of a principal plane, stated above, is therefore equivalent with the condition that there exist admissible values of λ, μ, ν which satisfy the equations (1) and for which $k \neq 0$. But, since these equations may be looked upon as linear homogeneous equations in λ, μ, and ν, it follows from Theorem 2, Chapter II (Section 22, page 38) that their coefficient determinant must vanish (since the trivial solution of these equations does not lead to admissible values of λ, μ, and ν); that is, we must have:

$$(2) \quad \begin{vmatrix} a_{11} - k & a_{12} & a_{13} \\ a_{12} & a_{22} - k & a_{23} \\ a_{13} & a_{23} & a_{33} - k \end{vmatrix} = 0.$$

This equation is a cubic in k; therefore it has 3 roots. To every root k^*, for which the rank of the corresponding matrix

$$(3) \quad \left\|\begin{matrix} a_{11} - k^* & a_{12} & a_{13} \\ a_{12} & a_{22} - k^* & a_{23} \\ a_{13} & a_{23} & a_{33} - k^* \end{matrix}\right\|$$

is 2, there corresponds (compare Corollary of Theorem 4, Chapter II, Section 25, page 42) a single infinitude of values of λ, μ, ν determining uniquely the ratios $\lambda : \mu : \nu$, and hence a single principal direction. If the rank of the matrix (3) is 1, the three equations (1)

* It should be clear that these equations can be derived, independently of the formula for sin θ, from the equation of the diametral plane by means of Theorem 7, Chapter IV, Section 44, page 78. The derivation followed in the text has the advantage of giving significance to the variable k.

are equivalent; there is therefore only one linear condition on λ, μ, and ν, so that an arbitrary admissible value may be assigned to one of these variables. Hence, to a value k^* of k, for which the rank of the matrix (3) is 1, there corresponds a single infinitude of principal directions. If there is a k^* which causes the rank of the matrix (3) to become 0, then the direction cosines λ, μ, ν are entirely unrestricted,† and hence every direction is a principal direction. In order to facilitate the statement of our results we introduce the following definition.

Definition XII. **The equation (2) is called the *discriminating equation* of the quadric surface Q; the *discriminating numbers* of the surface Q are the roots of the discriminating equation.**

Theorem 18. **Every quadric surface has three discriminating numbers; to each of these corresponds a single principal direction, a single infinitude of principal directions, or all directions, according as it gives the matrix (3) the rank 2, 1 or 0; with every discriminating number which is different from zero, there is associated a principal plane at finite distance.**

Remark. The direction cosines of a principal direction, associated with the discriminating number k_i will be denoted by λ_i, μ_i, and ν_i, $i = 1, 2, 3$; they are found by solving any two of the equations (1) for the ratios $\lambda_i : \mu_i : \nu_i$.

Since $q(x, y, z)$ is a homogeneous function of the second degree in x, y, z, we find by application of Euler's theorem (see footnote on page 161) that

$$\begin{aligned}2q(\lambda_i, \mu_i, \nu_i) &= \lambda_i q_1(\lambda_i, \mu_i, \nu_i) + \mu_i q_2(\lambda_i, \mu_i, \nu_i) + \nu_i q_3(\lambda_i, \mu_i, \nu_i) \\ &= 2\, k_i(\lambda_i^2 + \mu_i^2 + \nu_i^2) = 2\, k_i.\end{aligned}$$

This leads us to the following useful corollary.

Corollary. **If λ_i, μ_i, ν_i are the direction cosines of a principal direction‡ corresponding to the discriminating number k_i of the quadric**

† That such a situation may arise should be clear geometrically from the fact that in a sphere every plane through the center bisects the chords which are perpendicular to it, so that every direction is a principal direction. In an ellipsoid of revolution, every plane through the axis of revolution bisects the chords perpendicular to it, so that every direction perpendicular to this axis is a principal direction; this furnishes an example of a surface in which the direction cosines of a principal direction are subject to only one linear condition.

‡ It is to be understood here that more than one principal direction may be associated with a single discriminating number.

surface Q, then

$$q_1(\lambda_i, \mu_i, \nu_i) = 2k_i\lambda_i, \quad q_2(\lambda_i, \mu_i, \nu_i) = 2k_i\mu_i, \quad q_3(\lambda_i, \mu_i, \nu_j) = 2k_i\nu_i,$$
$$q(\lambda_i, \mu_i, \nu_i) = k_i, \quad i = 1, 2, 3.$$

89. The Discriminating Equation. We proceed now to a further discussion of the discriminating equation

$$(1)\quad \begin{vmatrix} a_{11} - k & a_{12} & a_{13} \\ a_{12} & a_{22} - k & a_{23} \\ a_{13} & a_{23} & a_{33} - k \end{vmatrix} = 0 \text{ of a quadric surface}\dagger$$

THEOREM 19. **A root k^* of the discriminating equation is a single, double, or triple root according as the rank of the matrix**

$$(2)\quad \left\| \begin{matrix} a_{11} - k^* & a_{12} & a_{12} \\ a_{12} & a_{22} - k^* & a_{23} \\ a_{13} & a_{23} & a_{33} - k^* \end{matrix} \right\|$$

is 2, 1 or 0; and conversely.

Proof. It follows from Theorem 19, Chapter I (Section 17, page 32) that, if the left-hand side of equation (1) is designated by $A(k)$ and its derivatives with respect to k by means of accents, then

$$A'(k) = \begin{vmatrix} -1 & 0 & 0 \\ a_{12} & a_{22} - k & a_{23} \\ a_{13} & a_{23} & a_{33} - k \end{vmatrix} + \begin{vmatrix} a_{11} - k & a_{12} & a_{13} \\ 0 & -1 & 0 \\ a_{13} & a_{23} & a_{33} - k \end{vmatrix}$$
$$+ \begin{vmatrix} a_{11} - k & a_{12} & a_{13} \\ a_{12} & a_{22} - k & a_{23} \\ 0 & 0 & -1 \end{vmatrix} = -\left\{ \begin{vmatrix} a_{22} - k & a_{23} \\ a_{23} & a_{33} - k \end{vmatrix} \right.$$
$$\left. + \begin{vmatrix} a_{11} - k & a_{13} \\ a_{13} & a_{33} - k \end{vmatrix} + \begin{vmatrix} a_{11} - k & a_{12} \\ a_{12} & a_{22} - k \end{vmatrix} \right\},$$

$$A''(k) = 2[(a_{11} - k) + (a_{22} - k) + (a_{33} - k)], \qquad A'''(k) = -6.$$

† It should be clear how equations analagous to the one written above can be formed for every symmetric square matrix $\|a_{ij}\|$, $i, j = 1, 2, \ldots, n$ of any order. Such an equation is usually called the characteristic equation of the matrix. The equation treated in the text is therefore the characteristic equation of the matrix $\mathbf{a}_3$. The characteristic equation of a matrix plays a very important role in the theory of matrices. Many of the properties developed in the text for the characteristic equation of the matrix $\mathbf{a}_3$ hold, with appropriate changes, for the characteristic equation of the general symmetric square matrix; and the methods of proof here used are readily adaptable to the general case.

Suppose now

(a) the rank of the matrix (2) is 0. Then clearly, $A(k^*) = 0$, $A'(k^*) = 0$, and $A''(k^*) = 0$; therefore k^* is a triple root of the equation $A(k) = 0$.

(b) the rank of the matrix (2) is 1. In this case $A(k^*) = 0$, $A'(k^*) = 0$, but $A''(k^*) \neq 0$. For, the rank of the matrix being 1, every two rowed-principal minor vanishes; that is, $(a_{ii} - k^*)(a_{jj} - k^*) = a_{ij}^2 \geqq 0$, $i, j = 1, 2, 3$; $i \neq j$. It follows from this that if two of the elements in the principal diagonal of (2) vanish, then all the elements outside the principal diagonal vanish also; hence, the remaining element in the principal diagonal must be different from zero and therefore $A''(k^*) \neq 0$. And it follows also from this relation that any two principal diagonal elements of (2) which do not vanish must be of the same sign, so that $A''(k^*)$ can vanish only, if each of its terms vanishes, which has been shown to contradict the hypothesis that the rank of the matrix (2) is 1. Consequently k^* is a double root of the equation (1).

(c) the rank of the matrix (2) is 2. In this case we can apply the Corollary of Theorem 7, Chapter II (Section 26, page 44); and we conclude that $A'(k^*) \neq 0$. It follows therefore that k^* is a simple root of equation (1).

The converse follows from the fact that the three cases in the hypothesis and in the conclusion both represent all possibilities. For example, if k^* is a double root of equation (1), the rank of the matrix (2) is 1; for if the rank were not 1, it would be 2 or 0, and k^* would therefore be a simple root or else a triple root of the equation.

If we expand the polynomial $A(k)$ according to Taylor's theorem, we find that

$$A(k) = A(0) + k\,A'(0) + \frac{k^2}{2} \cdot A''(0) + \frac{k^3}{6} \cdot A'''(0).$$

From the formulæ developed in the proof of the theorem, we find that

$A(0) = |a_{ij}|$, $i, j = 1, 2, 3$; that is, $A(0) = A_{44}$ (compare page 173),

$$A'(0) = -\left[\begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}\right],$$

$= -[\alpha_{11} + \alpha_{22} + \alpha_{33}]$, (compare page 184)

$A''(0) = 2(a_{11} + a_{22} + a_{33})$, $A'''(0) = -6$.

We shall now use the following abbreviations:

$$T_1 = a_{11} + a_{22} + a_{33}, \qquad T_2 = \alpha_{11} + \alpha_{22} + \alpha_{33}.$$

Our discussion yields then the following corollary.

COROLLARY 1. **The expanded form of the discriminating equation of the quadric surface Q is**

$$\text{(3)} \quad k^3 - T_1k^2 + T_2k - A_{44} = 0.$$

For convenience of future reference we record here also the following results obtained by applying Theorem 18, Chapter I (see Section 16, page 29; see also the Corollary of Theorem 5, Chapter II, Section 26, page 43) and Theorem 7, Chapter II (see Section 26, page 44) to the determinant A_{44}.

COROLLARY 2. **Between the value of the determinant A_{44}, the elements in its principal diagonal and the elements α_{ij} of its adjoint, the following relations hold:**

$$\alpha_{11}\alpha_{22} - \alpha_{12}^2 = a_{33}A_{44}, \quad \alpha_{22}\alpha_{33} - \alpha_{23}^2 = a_{11}A_{44}, \quad \alpha_{33}\alpha_{11} - \alpha_{13}^2 = a_{22}A_{44}.$$

COROLLARY 3. **If the determinant A_{44} vanishes, then those of its principal two-rowed minors which do not vanish are of like signs.**

THEOREM 20. **The discriminating numbers of a real quadric surface are real.**

Proof. We will show first that, if the coefficients a_{ij} in the equation of the surface are real, then the discriminating equation can not have a root which is a pure imaginary. Suppose that iq is a root of equation (1). Then, according to a well-known theorem of algebra, $-iq$ must also be a root of this equation; that is, we will have

$$\begin{vmatrix} a_{11}-iq & a_{12} & a_{13} \\ a_{12} & a_{22}-iq & a_{23} \\ a_{13} & a_{23} & a_{33}-iq \end{vmatrix} = 0, \text{ and } \begin{vmatrix} a_{11}+iq & a_{12} & a_{13} \\ a_{12} & a_{22}+iq & a_{23} \\ a_{13} & a_{23} & a_{33}+iq \end{vmatrix} = 0.$$

But the product of the two determinants on the left-hand sides of these equations must then also vanish; application of Theorem 16, Chapter I (Section 14, page 26) gives us therefore the further result

$$\begin{vmatrix} \sum_{k=1}^{3} a_{1k}^2 + q^2 & \sum_1^3 a_{1k}a_{2k} & \sum_1^3 a_{1k}a_{3k} \\ \sum_1^3 a_{1k}a_{2k} & \sum_1^3 a_{2k}^2 + q^2 & \sum_1^3 a_{2k}a_{3k} \\ \sum_1^3 a_{1k}a_{3k} & \sum_1^3 a_{2k}a_{3k} & \sum_1^3 a_{3k}^2 + q^2 \end{vmatrix} = 0.$$

This last equation is of the same general form as equation (1) and is obtained from it if we substitute $-q^2$ for k and $\sum_{k=1}^{3} a_{ik}a_{jk}$ for a_{ij}. Corollary 1 of Theorem 19, enables us to state therefore that the expanded form of this equation is

$$(4) \quad q^6 + S_1 q^4 + S_2 q^2 + S_3 = 0,$$

where

$$S_1 = \sum_{k=1}^{3} a_{1k}^2 + \sum_{k=1}^{3} a_{2k}^2 + \sum_{k=1}^{3} a_{3k}^2,$$

$$S_2 = \begin{vmatrix} \sum_1^3 a_{2k}^2 & \sum_1^3 a_{2k}a_{3k} \\ \sum_1^3 a_{2k}a_{3k} & \sum_1^3 a_{3k}^2 \end{vmatrix} + \begin{vmatrix} \sum_1^3 a_{1k}^2 & \sum_1^3 a_{1k}a_{3k} \\ \sum_1^3 a_{1k}a_{3k} & \sum_1^3 a_{3k}^2 \end{vmatrix}$$

$$+ \begin{vmatrix} \sum_1^3 a_{1k}^2 & \sum_1^3 a_{1k}a_{2k} \\ \sum_1^3 a_{1k}a_{2k} & \sum_1^3 a_{2k}^2 \end{vmatrix},$$

$$\text{and } S_3 = \begin{vmatrix} \sum_1^3 a_{1k}^2 & \sum_1^3 a_{1k}a_{2k} & \sum_1^3 a_{1k}a_{3k} \\ \sum_1^3 a_{1k}a_{2k} & \sum_1^3 a_{2k}^2 & \sum_1^3 a_{2k}a_{3k} \\ \sum_1^3 a_{1k}a_{3k} & \sum_1^3 a_{2k}a_{3k} & \sum_1^3 a_{3k}^2 \end{vmatrix}.$$

It should be clear that $S_3 = A_{44}^2$, and therefore that S_1 and S_3 are both non-negative. To show that the same thing is true of S_2, we observe that to each determinant in S_2 we can apply the Lemma preceding Theorem 14, Chapter III (Section 36, page 64); thus S_2 is transformed in to the sum of the squares of 9 two-rowed determinants. Consequently equation (4) has no negative coefficients and therefore, considered as a cubic in q^2, no solutions for which q^2 is positive. Therefore equation (2), the discriminating equation of the quadric surface, can have no root of the form iq, where q is real, unless $q = 0$.

Suppose next that equation (1) has a root of the form $p + iq$; then the new equation obtained from (1) by replacing a_{11}, a_{22}, a_{33}

by $a_{11} - p$, $a_{22} - p$, $a_{33} - p$ has the root iq. But the new equation is of the same form as equation (1) and therefore it can not have a root iq. Our theorem has therefore been proved.

THEOREM 21. **Not all the discriminating numbers of a quadric surface can be equal to zero.**

Proof. If all the roots of equation (3) were zero, then we would have $A_{44} = 0$, $T_2 = 0$, and $T_1 = 0$. But from the last two of these equations we could then conclude that

$$\begin{aligned} T_1^2 - 2\,T_2 &= (a_{11}^2 + a_{22}^2 + a_{33}^2 + 2\,a_{22}a_{33} + 2\,a_{33}a_{11} + 2\,a_{11}a_{22}) \\ &\quad - 2(a_{22}a_{33} - a_{23}^2) - 2(a_{33}a_{11} - a_{13}^2) - 2(a_{11}a_{22} - a_{12}^2) \\ &= a_{11}^2 + a_{22}^2 + a_{33}^2 + 2\,a_{23}^2 + 2\,a_{13}^2 + 2\,a_{12}^2 = 0 \end{aligned}$$

and from this it would follow that $a_{11} = a_{22} = a_{33} = a_{23} = a_{13} = a_{12} = 0$, which would mean that the equation of the surface contains no terms of the second degree.

COROLLARY. **The functions T_1 and T_2 can not both vanish; if $T_1 = 0$, then $T_2 < 0$.**

The first part of this corollary follows also from Theorem 20 (page 192), for, if $T_1 = T_2 = 0$, the discriminating equation reduces to $k^3 = A_{44}$, which has only one real root.

90. Principal Planes and Principal Directions. The results of the two preceding sections enable us to establish some further properties of principal planes and principal directions.

THEOREM 22. **Every quadric surface has at least one real principal plane at finite distance.**

This theorem is an immediate consequence of Theorems 18 (last part) and 21.

THEOREM 23. **The principal directions which correspond to two distinct discriminating numbers are mutually perpendicular.**

Proof. Let k_1 and k_2 be two discriminating numbers of the quadric surface and let $k_1 \neq k_2$. Then, in the notation of the Remark following Theorem 18 (see page 189) and in virtue of the Corollary of this theorem, we have

$$\begin{aligned} q_1(\lambda_1, \mu_1, \nu_1) &= k_1\lambda_1, \quad q_2(\lambda_1, \mu_1, \nu_1) = k_1\mu_1, \quad q_3(\lambda_1, \mu_1, \nu_1) = k_1\nu_1; \\ q_1(\lambda_2, \mu_2, \nu_2) &= k_2\lambda_2, \quad q_2(\lambda_2, \mu_2, \nu_2) = k_2\mu_2, \quad q_3(\lambda_2, \mu_2, \nu_2) = k_2\nu_2. \end{aligned}$$

Now the reader should have no difficulty in showing, by writing out the expressions in full, that

$$\lambda_2 q_1(\lambda_1, \mu_1, \nu_1) + \mu_2 q_2(\lambda_1, \mu_1, \nu_1) + \nu_2 q_3(\lambda_1, \mu_1, \nu_1) = \lambda_1 q_1(\lambda_2, \mu_2, \nu_2) + \mu_1 q_2(\lambda_2, \mu_2, \nu_2) + \nu_1 q_3(\lambda_2, \mu_2, \nu_2).$$

Therefore, if we multiply the equations of the first set written above by λ_2, μ_2, and ν_2 respectively and those of the second set by λ_1, μ_1, ν_1 respectively, we find that

$$(k_1 - k_2)(\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2) = 0.$$

Since we supposed that $k_1 \neq k_2$, we conclude that $\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0$, from which the theorem follows in virtue of Corollary 1 of Theorem 13, Chapter III, Section 36, page 64.

THEOREM 24. **If a quadric surface has three distinct discriminating numbers, then there exist three mutually perpendicular principal directions for the surface; if there are two distinct discriminating numbers, one principal direction is defined, and the second and third principal directions are any directions perpendicular to the first; if there is only one discriminating number, the principal directions are entirely arbitrary.**

This theorem follows from Theorems 18, 19, and 23 and the obvious facts that if the three discriminating numbers are distinct, each of them is a simple root of the discriminating equation; if there are two distinct discriminating numbers, one of them is a simple root and the other a double root; and if there is only one discriminating number it must be a triple root of the equation.

91. Exercises.

1. Determine for each of the following surfaces, whether or not an asymptotic cone exists; set up the equation of this cone in the cases in which one is present, and indicate whether the cone is proper or degenerate, real or imaginary:

(a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1;$

(b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1;$

(c) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1;$

(d) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2z;$

(e) $\dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 2x;$

(f) $\dfrac{x^2}{a^2} + \dfrac{z^2}{c^2} = 1;$

(g) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1;$

(h) $x^2 = 4pz.$

2. Discuss the asymptotic cone for the surfaces in parts (a), (b), (c), and (d) of Exercise 7, Section 86.

3. Determine the diametral planes of the surface

$$4x^2 - 6y^2 + 3z^2 + 2yz - 2zx - 4xy - 6x + 10y + 4z - 3 = 0$$

which correspond to the following directions:

(a) $\lambda : \mu : \nu = 1 : 1 : 1$, (b) $\lambda : \mu : \nu = 4 : -1 : 8$, (c) $\lambda : \mu : \nu = 6 : -2 : -3$.

4. Are there any directions with which no diametral plane of the surface in the preceding exercise is associated? Are there directions for which the associated diametral plane is not at finite distance?

5. Determine for each of the following surfaces the directions for which there is no diametral plane or no diametral plane at finite distance:

(a) $2x^2 + 20y^2 + 18z^2 - 12yz + 12xy + 22x + 6y - 2z - 5 = 0$;
(b) $2x^2 + 20y^2 + 18z^2 - 12yz + 12xy + 6x + 16y + 6z - 3 = 0$;
(c) $36x^2 + 4y^2 + z^2 - 4yz - 12zx + 24xy + 4x + 16y - 26z + 1 = 0$;
(d) $2x^2 - 7y^2 + 2z^2 - 10yz - 8zx - 10xy + 6x + 12y - 6z + 5 = 0$.

6. Prove that a quadric surface which has a single center (proper center or vertex) has a diametral plane at finite distance associated with every direction.

7. Prove that a quadric surface with a line of centers has a diametral plane at finite distance for every direction except one.

8. Prove that for a quadric surface with a plane of centers there exists an infinite number of directions with which no diametral plane at finite distance is associated.

9. Set up the discriminating equation for each of the following surfaces:

(a) $2x^2 - 7y^2 + 2z^2 - 10yz - 8zx - 10xy + 4x - 2y + 3z - 7 = 0$;
(b) $2x^2 + 2y^2 + 2z^2 + 4xz + 4y + 5 = 0$;
(c) $2x^2 + 2y^2 + 2z^2 + 2yz + 2zx + 2xy + 4x + 4y + 4z + 3 = 0$;
(d) $x^2 + 4y^2 + 9z^2 - 4xy - 12yz + 6zx - x + 2y + 5z = 0$.

10. Determine three principal directions for each of the following surfaces:

(a) $2x^2 + 2y^2 + 2z^2 + 2yz + 2zx + 2xy + 4x - 4y + 4z + 3 = 0$;
(b) $x^2 + y^2 + 2z^2 - 4xz + 2xy + 1 = 0$;
(c) $x^2 + y^2 - 2z^2 + 4yz + 4zx + 8xy - 6x + 5y - 4z + 6 = 0$;
(d) $x^2 + z^2 + 2xy + 2xz - 2yz - 2x + 4y - 4 = 0$;
(e) $13x^2 + 13y^2 + 10z^2 + 4yz + 4zx + 8xy - 3x - 4y + 2z - 6 = 0$;
(f) $x^2 + y^2 + z^2 - 4x + 6y + 2z - 5 = 0$.

CHAPTER VIII

CLASSIFICATION OF QUADRIC SURFACES

92. Invariants. The properties of quadric surfaces which were discussed in the preceding chapter, such as the existence of an asymptotic cone, of straight lines on the surface, of centers and of diametral planes, do not depend in any way on the frame of reference which is used; they are intrinsic properties of the surface. The algebraic magnitudes and relations involving the coefficients of the equations of the surfaces, by means of which these properties were characterized, must therefore be preserved when a new reference frame is introduced. This fact is expressed by the statement that these expressions and relations are invariant with respect to the transformation of coördinates which carry us over from one reference frame to another. Conversely, it is to be expected that an expression or a relation involving the coefficients of the equation of a surface which remains unchanged under such a transformation of coördinates, has an important bearing on the intrinsic geometrical properties of the surface. Indeed the search for such expressions and relations furnishes a method for the systematic study of these properties; it is therefore of fundamental importance in the entire field of Analytical Geometry. We shall undertake now to prove the existence of a number of such expressions and relations. But before doing so, we shall give an exact definition of the concept of invariance and we shall illustrate it by a few familiar examples.

DEFINITION I. **An expression involving the coefficients in the equation of a surface in Cartesian coördinates, and numbers which depend on these coefficients, are called *invariants of the surface* with respect to a transformation of coördinates which leads to another Cartesian reference frame, if they remain unchanged when the coefficients of the equation are replaced by the corresponding coefficients of the equation obtained from the given one by such a transformation of coördinates; relations between the coefficients which are preserved under such a transformation are called *invariant relations* with respect to the transformation.**

Examples.

1. The unsigned distance from the origin to the plane $ax + by + cz + d = 0$ is given by $\left|\dfrac{d}{\sqrt{a^2 + b^2 + c^2}}\right|$ (compare Corollary 1 of Theorem 7, Chapter IV, Section 44, page 79). This distance remains the same no matter what Cartesian reference frame is used, so long as the origin is not changed. Consequently, if we put

$$x = \lambda_1 x_1 + \lambda_2 y_1 + \lambda_3 z_1, \quad y = \mu_1 x_1 + \mu_2 y_1 + \mu_3 z_1, \quad z = \nu_1 x_1 + \nu_2 y_1 + \nu_3 z_1$$

in the equation $ax + by + cz + d = 0$ and if we suppose that the equation of the plane is thereby transformed into $a'x_1 + b'y_1 + c'z_1 + d' = 0$, then it must be true that $\left|\dfrac{d'}{\sqrt{a'^2 + b'^2 + c'^2}}\right| = \left|\dfrac{d}{\sqrt{a^2 + b^2 + c^2}}\right|$. If this is so, the expression $\left|\dfrac{d}{\sqrt{a^2 + b^2 + c^2}}\right|$ will be called an *invariant* of the plane with respect to the linear homogeneous transformation indicated above. To verify this fact, we observe that

$$a' = a\lambda_1 + b\mu_1 + c\nu_1, \quad b' = a\lambda_2 + b\mu_2 + c\nu_2, \quad c' = a\lambda_3 + b\mu_3 + c\nu_3, \quad d' = d.$$

Therefore

$$a'^2 + b'^2 + c'^2 = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)a^2 + (\mu_1^2 + \mu_2^2 + \mu_3^2)b^2 + (\nu_1^2 + \nu_2^2 + \nu_3^2) c^2 + 2(\mu_1\nu_1 + \mu_2\nu_2 + \mu_3\nu_3)bc + 2(\nu_1\lambda_1 + \nu_2\lambda_2 + \nu_3\lambda_3)ca + 2(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3)ab.$$

But it follows from Theorem 6, Chapter V (see Section 65, page 123) that the coefficients of a^2, b^2, and c^2 are each equal to unity and that the coefficients of bc, ca, and ab vanish; hence $a'^2 + b'^2 + c'^2 = a^2 + b^2 + c^2$. Thus we have proved that $\left|\dfrac{d}{\sqrt{a^2 + b^2 + c^2}}\right|$ is an invariant of the plane with respect to rotation of axes. Properly speaking this expression is an invariant with respect to rotation of axes of the configuration consisting of the plane *and* the origin.

2. The angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is independent of the reference frame that has been used to represent these planes. Therefore the expression for the cosine of this angle, given in Theorem 9, Chapter IV (Section 46, page 82), should be an invariant with respect to a transformation from one rectangular Cartesian system to another. Such a transformation is given by the equations

$$x = \lambda_1 x_1 + \lambda_2 y_1 + \lambda_3 z_1 + p, \quad y = \mu_1 x_1 + \mu_2 y_1 + \mu_3 z_1 + q, \quad z = \nu_1 x_1 + \nu_2 y_1 + \nu_3 z_1 + r.$$

(See Theorem 8, Chapter V, Section 66, page 126.)

If these expressions transform the equations of the given planes to

$$a_1'x_1 + b_1'y_1 + c_1'z_1 + d_1' = 0 \quad \text{and} \quad a_2'x_1 + b_2'y_1 + c_2'z_1 + d_2' = 0$$

we have

$$a_1' = a_1\lambda_1 + b_1\mu_1 + c_1\nu_1, \quad b_1' = a_1\lambda_2 + b_1\mu_2 + c_1\nu_2, \quad c_1' = a_1\lambda_3 + b_1\mu_3 + c_1\nu_3,$$
$$d_1' = a_1p + b_1q + c_1r,$$
$$a_2' = a_2\lambda_1 + b_2\mu_1 + c_2\nu_1, \quad b_2' = a_2\lambda_2 + b_2\mu_2 + c_2\nu_2, \quad c_2' = a_2\lambda_3 + b_2\mu_3 + c_2\nu_3,$$
$$d_2' = a_2p + b_2q + c_2r.$$

It follows from example 1 therefore that

$$a_1^2 + b_1^2 + c_1^2 = a_1'^2 + b_1'^2 + c_1'^2 \text{ and } a_2^2 + b_2^2 + c_2^2 = a_2'^2 + b_2'^2 + c_2'^2.$$

Moreover

$$a_1'a_2' + b_1'b_2' + c_1'c_2' = a_1a_2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + b_1b_2(\mu_1^2 + \mu_2^2 + \mu_3^2) + c_1c_2(\nu_1^2 + \nu_2^2 + \nu_3^2) + (a_1b_2 + a_2b_1)(\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3) + (b_1c_2 + b_2c_1)(\mu_1\nu_1 + \mu_2\nu_2 + \mu_3\nu_3) + (c_1a_2 + c_2a_1)(\nu_1\lambda_1 + \nu_2\lambda_2 + \nu_3\lambda_3) = a_1a_2 + b_1b_2 + c_1c_2.$$

Consequently

$$\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \times \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

is indeed an invariant with respect to the linear transformation of coördinates. In particular we notice that the relation $a_1a_2 + b_1b_2 + c_1c_2 = 0$ is invariant; the reader should see the geometric significance of this fact.

93. Invariants of a Quadric Surface with respect to Rotation and Translation of Axes. We proceed now to the following important theorems.

THEOREM 1. **The functions T_1, T_2, A_{44} and Δ of the coefficients of a quadric surface are invariant under translation of axes.**

Proof. Translation of axes is accomplished by means of the transformation $x = x' + p$, $y = y' + q$, $z = z' + r$ (see Theorem 2, Chapter V, Section 61, page 115). The equation of the quadric surface $Q(x, y, z) = 0$ with respect to the new reference frame is therefore $Q(x' + p, y' + q, z' + r) = 0$. But, in virtue of Section 75 and the notation introduced in Section 80, this equation may be written in the form:

$$(1) \quad Q(p, q, r) + x'Q_1(p, q, r) + y'Q_2(p, q, r) + z'Q_3(p, q, r) + q(x', y', z') = 0.$$

It follows from this that the coefficients of the second degree terms in the new equation of the surface are the same as those of the corresponding terms in the original equation; that is, if we differentiate between the new and the original equation by the use of a $'$,

$$a_{11}' = a_{11}, \quad a_{12}' = a_{12}, \quad a_{13}' = a_{13}, \quad a_{22}' = a_{22}, \quad a_{23}' = a_{23}, \quad a_{33}' = a_{33}.$$

Moreover, we see at once that

$$a_{14}' = \tfrac{1}{2} \cdot Q_1(p, q, r), \ a_{24}' = \tfrac{1}{2} \cdot Q_2(p, q, r) \ a_{34}' = \tfrac{1}{2} \cdot Q_3(p, q, r), \quad a_{44}' = Q(p, q, r).$$

Therefore (see Section 89, page 192)

$$T_1' = a_{11}' + a_{22}' + a_{33}' = a_{11} + a_{22} + a_{33} = T_1;$$

$$T_2' = \begin{vmatrix} a_{22}' & a_{23}' \\ a_{23}' & a_{33}' \end{vmatrix} + \begin{vmatrix} a_{11}' & a_{13}' \\ a_{13}' & a_{33}' \end{vmatrix} + \begin{vmatrix} a_{11}' & a_{12}' \\ a_{12}' & a_{22}' \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = T_2;$$

$$A_{44}' = \begin{vmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{12}' & a_{22}' & a_{23}' \\ a_{13}' & a_{23}' & a_{33}' \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = A_{44}.$$

It remains to show that $\Delta' = \Delta$. From what we have already proved, it follows that

$$\Delta' = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dfrac{Q_1(p, q, r)}{2} \\ a_{12} & a_{22} & a_{23} & \dfrac{Q_2(p, q, r)}{2} \\ a_{13} & a_{23} & a_{33} & \dfrac{Q_3(p, q, r)}{2} \\ \dfrac{Q_1}{2} & \dfrac{Q_2}{2} & \dfrac{Q_3}{2} & Q(p, q, r) \end{vmatrix}.$$

Since $\dfrac{Q_i(p, q, r)}{2} = a_{i1}p + a_{i2}q + a_{i3}r + a_{i4}$, $i = 1, 2, 3, 4$, and also since $2\,Q(p, q, r) = pQ_1(p, q, r) + qQ_2(p, q, r) + rQ_3(p, q, r) + Q_4(p, q, r)$ (see Corollary 1 of Theorem 4, Chapter VII, Section 81, page 162), it follows as in Section 84, page 172, that if to the last row of this determinant are added the products of the first three rows by $-p$, $-q$, and $-r$ respectively, and then to the last column, are added the products of the first three columns by $-p$, $-q$, $-r$ respectively, then this determinant reduces to the discriminant Δ of the surface. This completes the proof of our theorem.

Corollary 1. **The discriminating numbers of a quadric surface are invariant under translation of axes.**

This theorem follows from the fact that the coefficients of the discriminating equation, namely, 1, $-T_1$, T_2, and $-A_{44}$, are invariant under translation of axes.

Corollary 2. **The rank of the matrix a_3 is invariant under translation of axes.**

COROLLARY 3. **The rank of the matrix $\mathbf{a}_4$ is invariant under translation of axes.**

Proof. The proof of the invariance of Δ shows that the matrix $\mathbf{a}_4'$ is obtained from the matrix $\mathbf{a}_4$ by means of elementary transformations (see Definition XIV, Chapter I, Section 10, page 18); it follows therefore from Theorem 14, Chapter I, that these two matrices have the same rank.

THEOREM 2. **The functions T_1, T_2, and A_{44} are invariant under rotation of axes.**

Proof. The proof of this theorem and of the next could be made by the direct method followed in the proof of Theorem 1, which consists in first expressing the coefficients of the new equation in terms of those of the given equation and then substituting these expressions in the function whose invariance we wish to prove. But this method, besides being laborious, does not give us any further insight into the geometric meaning of the theorem. We shall therefore follow a method of proof which is apparently less direct and which may impress the reader as being rather sophisticated, but which has the merit, apart from greater elegance and brevity, of penetrating more deeply into the problem under consideration.

We consider the function $\bar{q}(x, y, z, k)$ defined as follows:

$$\bar{q}(x, y, z, k) = q(x, y, z) - k(x^2 + y^2 + z^2).$$

Let an arbitrary rotation of axes carry the function $q(x, y, z)$ over into the function $q'(x', y', z')$. Since the expression $x^2 + y^2 + z^2$ represents the square of the distance from the origin to the point (x, y, z), it is invariant under rotation of axes; that is, $x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$. Hence, if the same rotation of axes changes the function $\bar{q}(x, y, z, k)$ to $\bar{q}'(x', y', z', k)$, we have

$$\bar{q}'(x', y', z', k) = q'(x', y', z') - k(x'^2 + y'^2 + z'^2).$$

The equation $\bar{q}(x, y, z, k) = 0$, being homogeneous in x, y, and z, for every value of k, represents a quadric cone; this will be a degenerate quadric cone (that is, a pair of planes) if and only if k has such a value k^* that the determinant

$$\begin{vmatrix} a_{11} - k^* & a_{12} & a_{13} \\ a_{12} & a_{22} - k^* & a_{23} \\ a_{13} & a_{23} & a_{33} - k^* \end{vmatrix}$$

vanishes. If this determinant vanishes, the equation $\bar{q}(x, y, z, k^*) = 0$ represents a pair of planes; therefore the equation $\bar{q}'(x', y', z', k^*) = 0$ represents a pair of planes and consequently the value of the determinant $\begin{vmatrix} a_{11}' - k^* & a_{12}' & a_{13}' \\ a_{12}' & a_{22}' - k^* & a_{23}' \\ a_{13}' & a_{23}' & a_{33}' - k^* \end{vmatrix}$ is also equal to zero. And it should be clear that the same argument holds in the opposite direction. From this we conclude, in view of Corollary 1 of Theorem 19, Chapter VII (see Section 89, page 192), that the two equations

$$k^3 - T_1k^2 + T_2k - A_{44} = 0 \text{ and } k^3 - T_1'k^2 + T_2'k - A_{44}' = 0$$

have the same roots, that is, $T_1 = T_1'$, $T_2 = T_2'$, $A_{44} = A_{44}'$; thus our theorem is proved.

If r_3, the rank of the matrix $\mathbf{a}_3$, is equal to 3, $A_{44} \neq 0$; therefore $A_{44}' \neq 0$ and $r_3' = 3$. If $r_3 = 2$, so that $A_{44} = 0$, then $A_{44}' = 0$ and $r_3' < 3$. If $r_3 = 1$, all the two-rowed minors of $\mathbf{a}_3$ vanish and therefore $T_2 = 0$; hence, it follows from Theorem 2, that $A_{44}' = 0$ and $T_2' = 0$, and thence by use of the Corollary of Theorem 7, Chapter II (Section 26, page 44) that $r_3' < 2$. If $r_3 = 0$, the function $Q(x, y, z)$ is of the first degree; therefore the function $Q'(x', y', z')$ is also of the first degree (compare Corollary 1 of Theorem 8, Chapter V, Section 66, page 126) and $r_3' = 0$. But this entire argument can be applied equally well to the transformation which carries Q' back into Q. It follows therefore that if $r_3' = 2$, then $r_3 < 3$; if $r_3' = 1$, then $r_3 < 2$; and if $r_3' = 0$, then $r_3 < 1$. We have therefore obtained the following important corollary.

Corollary. **The rank of the matrix $\mathbf{a}_3$ is invariant with respect to rotation of axes.**

94. Invariance of the Discriminant of a Quadric Surface with respect to Rotation.

We shall begin by proving the following theorem.

Theorem 3. **The singularity of a quadric surface is not affected by rotation of axes.**

Proof. In view of Definition V of Chapter VII (Section 82, page 166) the statement of this theorem is equivalent to the statement that if $\Delta = 0$, then $\Delta' = 0$, where Δ' is formed from

the equation obtained from $Q(x, y, z) = 0$ by rotation of axes; or again, to the statement that if $r_4 < 4$, then $r_4' < 4$, where r_4' designates the rank of the matrix $\mathbf{a}_4'$ formed from this same equation. All the cases in which $r_4 < 4$ have been specified geometrically in Remark 1, following Theorem 14, Chapter VII (Section 85, page 179), excepting the case in which $r_4 = 3$ and $r_3 = 1$. In this case, we know on the basis of the Corollary to Theorem 2 (Section 93) that $r_3' = 1$ and hence, in view of the discussion preceding Theorem 14 of Chapter VII, that r_4' can not exceed 3. We conclude therefore that in every case in which $r_4 < 4$, we must also have $r_4' < 4$. Our theorem is therefore proved.

THEOREM 4. **The discriminant of a quadric surface is invariant with respect to rotation of axes.**

Proof. The method of proof is similar to that used in the proof of Theorem 2. We consider now the auxiliary function

$$\overline{Q}(x, y, z, k) = Q(x, y, z) - k(x^2 + y^2 + z^2 + 1).$$

Rotation of axes will carry this function over into

$$\overline{Q'}(x', y', z', k) = Q'(x', y', z') - k(x'^2 + y'^2 + z'^2 + 1).$$

A value k^* of k for which the locus of the equation $\overline{Q}(x, y, z, k) = 0$ is singular will, in virtue of the preceding theorem, also be a value of k for which the surface $\overline{Q'}(x', y', z', k) = 0$ is singular. Hence the roots of the equation

$$\Delta(k) = \begin{vmatrix} a_{11} - k & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} - k & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} - k & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} - k \end{vmatrix} = 0$$

will also be roots of the equation

$$\begin{vmatrix} a_{11}' - k & a_{12}' & a_{13}' & a_{14}' \\ a_{12}' & a_{22}' - k & a_{23}' & a_{24}' \\ a_{13}' & a_{23}' & a_{33}' - k & a_{34}' \\ a_{14}' & a_{24}' & a_{34}' & a_{44}' - k \end{vmatrix} = 0,$$

and vice versa. These equations are therefore equivalent. Now it should be obvious that they have the form

$k^4 + \ldots + \Delta = 0$ and $k^4 + \ldots + \Delta' = 0$; and therefore that $\Delta = \Delta'$. This proves our theorem.

It will be worth while to consider in further detail the equation $\Delta(k) = 0$, which is quite similar in form to the discriminating equation considered in Section 89. If we use again Theorem 19 of Chapter I (see Section 17, page 32), we find

$$\Delta'(k) = -\begin{vmatrix} a_{22}-k & a_{23} & a_{24} \\ a_{23} & a_{33}-k & a_{34} \\ a_{24} & a_{34} & a_{44}-k \end{vmatrix} - \begin{vmatrix} a_{11}-k & a_{13} & a_{14} \\ a_{13} & a_{33}-k & a_{34} \\ a_{14} & a_{34} & a_{44}-k \end{vmatrix}$$
$$- \begin{vmatrix} a_{11}-k & a_{12} & a_{14} \\ a_{12} & a_{22}-k & a_{24} \\ a_{14} & a_{24} & a_{44}-k \end{vmatrix} - \begin{vmatrix} a_{11}-k & a_{12} & a_{13} \\ a_{12} & a_{22}-k & a_{23} \\ a_{13} & a_{23} & a_{33}-k \end{vmatrix},$$

$$\Delta''(k) = 2\left[\begin{vmatrix} a_{33}-k & a_{34} \\ a_{34} & a_{44}-k \end{vmatrix} + \begin{vmatrix} a_{22}-k & a_{24} \\ a_{24} & a_{44}-k \end{vmatrix} + \begin{vmatrix} a_{11}-k & a_{14} \\ a_{14} & a_{44}-k \end{vmatrix}\right.$$
$$\left. + \begin{vmatrix} a_{22}-k & a_{23} \\ a_{23} & a_{33}-k \end{vmatrix} + \begin{vmatrix} a_{11}-k & a_{13} \\ a_{13} & a_{33}-k \end{vmatrix} + \begin{vmatrix} a_{11}-k & a_{12} \\ a_{12} & a_{22}-k \end{vmatrix}\right],$$

$$\Delta'''(k) = -6\,[(a_{11}-k) + (a_{22}-k) + (a_{33}-k) + (a_{44}-k)],$$
$$\Delta''''(k) = 24.$$

Since, moreover, $\Delta(k) = \Delta(0) + \Delta'(0) \times k + \Delta''(0) \times \frac{k^2}{2}$

$+\Delta'''(0) \times \frac{k^3}{3!} + \Delta''''(0) \times \frac{k^4}{4!}$, the equation $\Delta(k) = 0$

can be written in the form

$$k^4 - D_1k^3 + D_2k^2 - D_3k + \Delta = 0,$$

where $D_1 = -\dfrac{\Delta'''(0)}{6} = a_{11} + a_{22} + a_{33} + a_{44}$,

$$D_2 = \frac{\Delta''(0)}{2} = \sum_{\substack{i,j=1\\ i<j}}^{4} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ij} & a_{jj} \end{vmatrix},$$

and $$D_3 = -\Delta'(0) = \sum_{\substack{i,j,k=1\\ i<j<k}}^{3} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ij} & a_{jj} & a_{jk} \\ a_{ik} & a_{jk} & a_{kk} \end{vmatrix}.$$

It will be observed that D_1, D_2, and D_3 are respectively the sums of the one-rowed, the two-rowed, and the three-rowed principal minors of the discriminant Δ.

We have now the following Corollary of Theorem 4.

Corollary. **The sums of the one-rowed, of the two-rowed, and of the three-rowed principal minors of the discriminant of a quadric surface are invariant with respect to rotation of axes.**

It follows moreover from Theorem 4 that, if $r_4 = 4$, then $r_4' = 4$; and that if $r_4 = 3$, then $r_4' < 4$. If $r_4 = 2$, all the three-rowed minors of $\mathbf{a}_4$ vanish and therefore $D_3 = 0$; the corollary enables us then to conclude that $D_3' = 0$ and the Corollary of Theorem 7, Chapter II (Section 26, page 44) establishes then the fact that $r_4' < 3$. Similarly it can be shown* that if $r_4 = 1$, then $r_4' < 2$; and it should be clear that if $r_4 = 0$, then $r_4' = 0$. Moreover the argument can be made equally well from the rank of Δ' to that of Δ. We have therefore obtained the further result, stated in the following theorem.

THEOREM 5. **The rank of the matrix $\mathbf{a}_4$ is invariant with respect to rotation of axes.**

The results obtained in Sections 93 and 94 may be summarized in the following statement:

> **The values of the expressions Δ, A_{44}, T_1, T_2 and the ranks of the matrices $\mathbf{a}_3$ and $\mathbf{a}_4$ are invariant with respect to translation and rotation of axes; the expressions D_1, D_2, and D_3 are invariant under rotation of axes.**

95. Exercises.

1. Prove that the condition under which three planes have a single point in common is invariant with respect to translation of axes, and also with respect to rotation of axes.

2. Prove that the distance from the plane $ax + by + cz + d = 0$ to the point $P(x_1, y_1, z_1)$ is invariant with respect to translation and rotation of axes.

3. Show that for the surface $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1 = 0$, the sum D_3 of the three-rowed principal minors of the discriminant is not invariant with respect to translation of axes.

4. Show that for the surface $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$, the sum D_3 is invariant with respect to translation of axes; also that the sum D_2 of the two-rowed principal minors of the discriminant and the sum D_1 of its one-rowed principal minors are not invariant with respect to this transformation of coördinates.

5. Show that for the surface $x^2 = a^2$, the sums D_3 and D_2 are invariant with respect to translation of axes; also that the sum D_1 is not invariant.

6. Prove that if the axes are translated to the new origin $P(\alpha, \beta, \gamma)$, the sum D_3' for the new equation is equal to D_3 plus multiples of three-rowed minors of the matrix $\mathbf{b}$ (compare Section 85, page 178).

7. Prove that under the conditions of Exercise 6, the sum D_2' for the new equation is equal to D_2 plus multiples of two-rowed minors of the matrix $\mathbf{b}$.

* See Appendix, IV, p. 297.

96. Two Planes. We have already met a number of instances in which the equation $Q(x, y, z) = 0$ represents two planes. In the present section we undertake a more detailed study of these cases; and we begin with the following theorem.

THEOREM 6. **The necessary and sufficient condition that a quadric surface consist of two planes is that the rank of r_4 of the discriminant matrix $\mathbf{a}_4$ be less than 3.**

Proof. The sufficiency of this condition has already been proved in Corollary 2 of Theorem 12, Chapter VII (Section 84, page 175). To prove the necessity of the condition, we observe that if the locus of the equation $Q(x, y, z) = 0$ consists of two planes then, by the argument made in the proof of this corollary, the function $Q(x, y, z)$ must be factorable in two linear factors; that is, there must exist numbers a, b, c, d and a_1, b_1, c_1, d_1 such that

$$Q(x, y, z) = (ax + by + cz + d)\,(a_1x + b_1y + c_1z + d_1).$$

If this is the case, the coefficients of the function Q can be expressed as follows:

$$\begin{aligned} &a_{11} = aa_1, \quad a_{22} = bb_1, \quad a_{33} = cc_1, \quad a_{44} = dd_1, \quad 2\,a_{12} = ab_1 + a_1b, \\ &\quad 2\,a_{13} = ac_1 + a_1c, \quad 2\,a_{14} = ad_1 + a_1d, \quad 2\,a_{23} = bc_1 + b_1c, \\ &\quad 2\,a_{24} = bd_1 + b_1d, \quad 2\,a_{34} = cd_1 + c_1d. \end{aligned}$$

Consequently the discriminant Δ is given by the equation

$$16\,\Delta = \begin{vmatrix} 2\,aa_1 & ab_1 + a_1b & ac_1 + a_1c & ad_1 + a_1d \\ ab_1 + a_1b & 2\,bb_1 & bc_1 + b_1c & bd_1 + b_1d \\ ac_1 + a_1c & bc_1 + b_1c & 2\,cc_1 & cd_1 + c_1d \\ ad_1 + a_1d & bd_1 + b_1d & cd_1 + c_1d & 2\,dd_1 \end{vmatrix}.$$

It is not difficult to show that this determinant and its three-rowed principal minors vanish (the details of this proof will be found in Appendix, V, page 298). But since the matrix of this determinant is symmetric, we can then conclude by use of Theorem 6, Chapter II (see Section 26, page 43) that the rank of the matrix $\mathbf{a}_4$ is less than 3.

It was proved in Corollary 3 of Theorem 14, Chapter VII (Section 85, page 180), that if $r_4 = r_3 = 2$, the locus of the equation $Q(x, y, z) = 0$ consists of two intersecting planes. We shall now determine the equations of these planes.

Since $r_3 = 2$, it follows that at least one of the two-rowed principal minors of the matrix $\mathbf{a}_3$ is different from zero; let us suppose that $\alpha_{33} = a_{11}a_{22} - a_{12}^2 \neq 0$. From Theorems 12 and 13 of Chapter I (see Section 7, page 13), we derive then the following equalities:

$$\alpha_{13}a_{11} + \alpha_{23}a_{12} + \alpha_{33}a_{13} = 0, \quad \alpha_{13}a_{12} + \alpha_{23}a_{22} + \alpha_{33}a_{23} = 0,$$
$$\alpha_{13}a_{13} + \alpha_{23}a_{23} + \alpha_{33}a_{33} = 0, \quad \alpha_{13}a_{14} + \alpha_{23}a_{24} + \alpha_{33}a_{34} = 0.$$

And if we denote by β_{ij} the cofactors of the elements a_{ij} in the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{14} & a_{24} & a_{34} \end{vmatrix}$, we have also (notice that $\beta_{34} = \alpha_{33} \neq 0$) the equalities

$$\beta_{13}a_{11} + \beta_{23}a_{12} + \beta_{34}a_{14} = 0, \quad \beta_{13}a_{12} + \beta_{23}a_{22} + \beta_{34}a_{24} = 0,$$
$$\beta_{13}a_{13} + \beta_{23}a_{23} + \beta_{34}a_{34} = 0, \quad \beta_{13}a_{14} + \beta_{23}a_{24} + \beta_{34}a_{44} = 0.$$

If we multiply the equalities of the first of these sets by x, y, z, and 1 respectively and add, we obtain $\alpha_{13}Q_1 + \alpha_{23}Q_2 + \alpha_{33}Q_3 = 0$; similarly, we find from the second set the result $\beta_{13}Q_1 + \beta_{23}Q_2 + \beta_{34}Q_4 = 0$. And from these equations we conclude (remembering that $\beta_{34} = \alpha_{33} \neq 0$), that

$$2\,\alpha_{33}Q(x, y, z) = \alpha_{33}(xQ_1 + yQ_2 + zQ_3 + Q_4) = (\alpha_{33}x - \alpha_{13}z - \beta_{13})Q_1 + (\alpha_{33}y - \alpha_{23}z - \beta_{23})Q_2.$$

But, $\alpha_{33}y - \alpha_{23}z - \beta_{23} = (a_{11}a_{22} - a_{12}^2)y + (a_{11}a_{23} - a_{12}a_{13})z + (a_{11}a_{24} - a_{12}a_{14}) = a_{11}Q_2 - a_{12}Q_1$;

and $\alpha_{33}x - \alpha_{13}z - \beta_{13} = (a_{11}a_{22} - a_{12}^2)x + (a_{22}a_{13} - a_{12}a_{23})z + (a_{22}a_{14} - a_{12}a_{24}) = a_{22}Q_1 - a_{12}Q_2$.

Therefore, $2\,\alpha_{33}Q(x, y, z) = (a_{11}Q_2 - a_{12}Q_1)Q_2 + (a_{22}Q_1 - a_{12}Q_2)Q_1 = a_{11}Q_2^2 - 2\,a_{12}Q_1Q_2 + a_{22}Q_1^2$.

Since the discriminant of this quadratic function of Q_1 and Q_2 is equal to $-4\,\alpha_{33}$ and is therefore different from zero, we conclude that the equation $Q(x, y, z) = 0$ is equivalent to the two linear equations $Q_2 - \lambda Q_1 = 0$ and $Q_2 - \mu Q_1 = 0$, where λ and μ are the distinct roots of the quadratic equation $a_{11}t^2 - 2\,a_{12}t + a_{22} = 0$. The two planes represented by the equation $Q = 0$ in this case are therefore distinct planes through the line of intersection of the planes $Q_1 = 0$ and $Q_2 = 0$ (compare page 180). It follows moreover that if $\alpha_{33} > 0$, the two roots of the quadratic equation $a_{11}t^2 - 2\,a_{12}t$

$+ a_{22} = 0$ are complex, whereas they are real if $\alpha_{33} < 0$. Finally, we observe that the sign of α_{33} is the same as that of the invariant T_2, which is equal to the sum of the two-rowed principal minors of the matrix $\mathbf{a}_3$, in virtue of Theorem 7 of Chapter II (see Section 26, page 44). We summarize the result in a theorem.

THEOREM 7. **If the ranks of the matrices $\mathbf{a}_4$ and $\mathbf{a}_3$ are both 2, the locus of the equation $Q(x, y, z) = 0$ consists of two planes; these planes are real if the invariant T_2 is negative, and imaginary if T_2 is positive.**

We consider next the case in which $r_4 = 2$ and $r_3 = 1$; it was shown on page 178 that in this case the rank of the matrix $\mathbf{b}$ is also equal to 1. Consequently the three rows of this matrix are proportional; and if we suppose that $a_{11} \neq 0$,* we can write $Q_2 = \dfrac{a_{12}Q_1}{a_{11}}$ and $Q_3 = \dfrac{a_{13}Q_1}{a_{11}}$. Moreover $Q_4 = q_4 + 2\ a_{44} = \dfrac{a_{14}q_1}{a_{11}}$ $+ 2\ a_{44} = \dfrac{a_{14}}{a_{11}} \times (Q_1 - 2\ a_{14}) + 2\ a_{44} = \dfrac{a_{14}Q_1}{a_{11}} - \dfrac{2(a_{14}^2 - a_{11}a_{44})}{a_{11}}$.

Therefore

$$2\,Q(x, y, z) = xQ_1 + yQ_2 + zQ_3 + Q_4 = Q_1\left(x + \frac{a_{12}y}{a_{11}} + \frac{a_{13}z}{a_{11}} + \frac{a_{14}}{a_{11}}\right) - \frac{2(a_{14}^2 - a_{11}a_{44})}{a_{11}},$$

and the equation $Q(x, y, z) = 0$ is equivalent to the equation

$$Q_1^2 = 2(a_{14}^2 - a_{11}a_{44}).$$

It is shown in Appendix, VI (page 299) that if $r_4 = 2$, the two-rowed principal minors of the matrix $\mathbf{a}_4$ can not all vanish and that those which do not vanish all have the same sign and therefore the sign of their sum D_2. Since $r_3 = 1$, all two-rowed principal minors of $\mathbf{a}_3$ vanish; the rank of $\mathbf{b}$ being 1, the other two-rowed principal minors of $\mathbf{a}_4$ differ by a factor and are therefore different from zero and of the same sign of D_2. We conclude that the locus of the equation $Q(x, y, z) = 0$ consists of two distinct parallel planes, whose equations are $Q_1 = \pm\sqrt{2(a_{14}^2 - a_{11}a_{44})}$; these planes will be real or imaginary according as D_2 is negative or positive.

* If all the elements in the principal diagonal of $\mathbf{a}_3$ were zero, it would follow since in this case $\alpha_{11} = \alpha_{22} = \alpha_{33} = 0$, that the elements a_{12}, a_{23}, and a_{13} also vanish, so that the rank r_3 of $\mathbf{a}_3$ would be zero; all the non-vanishing elements of the principal diagonal have the sign of the invariant T_1.

Finally, the same discussion shows that if $r_4 = r_3 = 1$, then the equation $Q(x, y, z) = 0$ reduces to the form $Q_1^2 = 0$, so that its locus consists of two coincident planes (compare Corollary 3 of Theorem 14, Section 85, page 180).

THEOREM 8. **If the rank of the matrix $\mathbf{a}_4$ is 2 and the rank of the matrix $\mathbf{a}_3$ is 1, the locus of the equation $Q(x, y, z) = 0$ consists of a pair of parallel planes; these planes are real or imaginary according as the sum D_2 of the two-rowed principal minors of the matrix $\mathbf{a}_4$ is negative or positive. If the ranks of the matrices $\mathbf{a}_4$ and $\mathbf{a}_3$ are both 1, the locus of the equation $Q(x, y, z) = 0$ consists of the plane $Q_1(x, y, z) = 0$, counted doubly.**

Remark. Since the hypotheses of Theorems 7 and 8 exhaust all the possibilities as to the ranks of the matrices $\mathbf{a}_4$ and $\mathbf{a}_3$, subject to the condition of Theorem 6 that r_4 must be less than 3; and since the conclusions of these two theorems include all the possible relative positions of two planes, it follows that the converse of each of these theorems also holds; that is, if a quadric surface consists of two real intersecting planes, two imaginary intersecting planes, two real parallel planes, two imaginary parallel planes, or two coincident planes, the ranks of the matrices $\mathbf{a}_4$ and $\mathbf{a}_3$ are 2 and $2(T_2 < 0)$, 2 and $2(T_2 > 0)$, 2 and $1(D_2 < 0)$, 2 and 1 $(D_2 > 0)$, 1 and 1 respectively.

We state some further consequences of our discussion.

COROLLARY 1. **If the rank of the matrix $\mathbf{a}_3$ is 2, the locus of the equation $q(x, y, z) = 0$ consists of a pair of intersecting planes, whose line of intersection passes through the origin; if the rank of this matrix is 1, the locus is a pair of coincident planes through the origin.**

COROLLARY 2. **A function $Q(x, y, z)$ of the second degree is factorable into two linear functions of x, y, and z with real or complex coefficients if and only if the rank of its discriminant matrix is less than 3; it is the square of a linear function of x, y, and z with real or complex coefficients if and only if the rank of its discriminant matrix is 1.**

COROLLARY 3. **A homogeneous function $q(x, y, z)$ of the second degree in x, y, and z is factorable into two linear homogeneous functions of x, y, z with real or complex coefficients if and only if the rank of the matrix $\mathbf{a}_3$ is less than 3; it is the square of a linear homogeneous function of x, y, and z with real or complex coefficients if and only if the rank of this matrix is 1.**

Corollaries 2 and 3 are obviously restatements in algebraic form of the results formulated in Theorems 6, 7, and 8.

97. Translation of Axes to the Center of a Quadric Surface. If a quadric surface has a center, its equation is materially simplified when the axes are translated to the center as origin. For it should be obvious from the definition of a center that the surface is symmetric with respect to such a point (compare Definition VIII of Chapter VII, Section 85, page 176 and the first footnote on page 137); therefore, if a, b, c are the coördinates of a point in the new reference frame, then $Q(-a, -b, -c)$ must vanish whenever $Q(a, b, c)$ vanishes. Consequently $Q(a, b, c) - Q(-a, -b, -c) = 0$ for all sets of numbers a, b, c for which $Q(a, b, c) = 0$. But $Q(a, b, c) - Q(-a, -b, -c) = 2(a_{14}a + a_{24}b + a_{34}c)$; if this linear function is to vanish for all sets of values for which the quadratic function $Q(a, b, c)$ vanishes, then $a_{14} = a_{24} = a_{34} = 0$. Consequently the equation of a quadric surface referred to a reference frame whose origin is a center of the surface does not have any first degree terms.

We shall now reach this result in another way, which will disclose some further properties. The translation of axes to the point (a, b, c) as origin is accomplished by means of the equations of transformation

$$x = x' + a,\ y = y' + b,\ z = z' + c.$$

The equation of the surface $Q(x, y, z) = 0$ with reference to the new system of coördinates is therefore

$$\begin{aligned} Q'(x', y', z') = Q(x'+a, y'+b, z'+c) = q(x', y', z') + x'Q_1(a, b, c) \\ + y'Q_2(a, b, c) + z'Q_3(a, b, c) + Q(a, b, c) = 0, \end{aligned}$$

(compare Section 93, formula (1), page 199).

But, if a, b, c are the coördinates of a center of the surface, $Q_1(a, b, c) = Q_2(a, b, c) = Q_3(a, b, c) = 0$ (see Theorem 13, Chapter VII, Section 85, page 177); in this case the equation of the surface reduces therefore to the form

$$q(x', y', z') + Q(a, b, c) = 0$$

and this equation is free from terms of the first degree in x', y', and z'.

We observe, moreover, (1) that the second degree terms in the new equation have the same coefficients as the corresponding terms of the original equation; and (2) that the constant term $Q(a, b, c)$ is equal to $\frac{1}{2}[aQ_1(a, b, c) + bQ_2(a, b, c) + cQ_3(a, b, c)$

$+Q_4(a, b, c)] = \frac{1}{2}Q_4(a, b, c) = a_{14}a + a_{24}b + a_{34}c + a_{44}$. Furthermore the discriminant of the simplified equation is, in virtue of Theorem 1 (Section 93, page 199) equal to the discriminant Δ of the original equation; on the other hand it is equal to

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{12} & a_{22} & a_{23} & 0 \\ a_{13} & a_{23} & a_{33} & 0 \\ 0 & 0 & 0 & \dfrac{Q_4}{2} \end{vmatrix} = A_{44} \cdot \frac{Q_4}{2}.$$

Therefore $\Delta = A_{44} \cdot \dfrac{Q_4}{2}$; if the surface has a unique center, and in no other case, $A_{44} \neq 0$, so that the constant term in the reduced equation can then also be put in the form $\dfrac{\Delta}{A_{44}}$. We summarize these results as follows.

THEOREM 9. **If the quadric surface Q has a center at the point (a, b, c), its equation in a reference frame whose axes are parallel to the original axes and whose origin is at the center, has the form $Q'(x', y', z') = q(x', y', z') + a_{44}' = 0$, where $a_{44}' = Q(a, b, c) = \dfrac{Q_4(a, b, c)}{2}$; if (a, b, c) is the only center of the surface, we have, moreover, $a_{44}' = \dfrac{\Delta}{A_{44}}$.**

98. Rotation of Axes to the Principal Directions of a Quadric Surface. It was proved in Theorem 24 of Chapter VII (Section 90, page 195) that for every quadric surface there exist three mutually perpendicular principal directions; under some conditions these directions can be determined in one and only one way; under other conditions they can be determined in more than one way. We will suppose now that for the quadric surface $Q(x, y, z) = 0$ three mutually perpendicular principal directions are given by the three sets of direction cosines λ_1, μ_1, ν_1; λ_2, μ_2, ν_2, and λ_3, μ_3, ν_3; and it is our purpose to determine the equation $Q'(x', y', z') = 0$ of the surface when it is referred to a reference frame whose origin coincides with the origin of the original frame, but whose axes are in these principal directions.

According to Theorem 5 of Chapter V (see Section 63, page 121), the transformation is carried out by means of the substitution:

$$x = \lambda_1 x' + \lambda_2 y' + \lambda_3 z', \quad y = \mu_1 x' + \mu_2 y' + \mu_3 z', \quad z = \nu_1 x' + \nu_2 y' + \nu_3 z'.$$

We have therefore

$$Q'(x', y', z') = Q(\lambda_1 x' + \lambda_2 y' + \lambda_3 z', \mu_1 x' + \mu_2 y' + \mu_3 z', \nu_1 x' + \nu_2 y' + \nu_3 z') = q(\lambda_1 x' + \lambda_2 y' + \lambda_3 z', \mu_1 x' + \mu_2 y' + \mu_3 z', \nu_1 x' + \nu_2 y' + \nu_3 z') + 2\, a_{14}(\lambda_1 x' + \lambda_2 y' + \lambda_3 z') + 2\, a_{24}(\mu_1 x' + \mu_2 y' + \mu_3 z') + 2\, a_{34}(\nu_1 x' + \nu_2 y' + \nu_3 z') + a_{44}.$$

Since $q(x, y, z)$ is a homogeneous function of the second degree in x, y, and z, and since the expressions which have been substituted for these variables are linear and homogeneous in x', y', and z', it should be clear that the function $q(\lambda_1 x' + \cdot\cdot\cdot, \mu_1 x' + \cdot\cdot\cdot, \nu_1 x' + \cdot\cdot\cdot)$ which constitutes the first term in the new equation is homogeneous and of the second degree in x', y', and z'. The terms of degree less than 2 in the new equation can be determined readily; if we write that part of the new equation in the form $2\, a_{14}'x' + 2\, a_{24}'y' + 2\, a_{34}'z' + a_{44}'$, we find

$$a_{14}' = a_{14}\lambda_1 + a_{24}\mu_1 + a_{34}\nu_1 = \frac{q_4(\lambda_1, \mu_1, \nu_1)}{2},$$

$$a_{24}' = a_{14}\lambda_2 + a_{24}\mu_2 + a_{34}\nu_2 = \frac{q_4(\lambda_2, \mu_2, \nu_2)}{2},$$

$$a_{34}' = a_{14}\lambda_3 + a_{24}\mu_3 + a_{34}\nu_3 = \frac{q_4(\lambda_3, \mu_3, \nu_3)}{2}, \qquad a_{44}' = a_{44}.$$

It remains now to determine the coefficients of the second degree terms in the new equation. For this purpose we expand $q(\lambda_1 x' + \cdot\cdot\cdot, \mu_1 x' + \cdot\cdot\cdot, \nu_1 x' + \cdot\cdot\cdot)$ by Taylor's theorem (compare Sections 75 and 80). First we look upon $\lambda_1 x' + \lambda_2 y'$, $\mu_1 x' + \mu_2 y'$ and $\nu_1 x' + \nu_2 y'$ as the (temporarily) fixed values of the variables in the function $q(x, y, z)$ and upon $\lambda_3 z'$, $\mu_3 z'$, and $\nu_3 z'$ as their increments. We find then

$$q(\lambda_1 x' + \lambda_2 y' + \lambda_3 z', \ \mu_1 x' + \mu_2 y' + \mu_3 z', \ \nu_1 x' + \nu_2 y' + \nu_3 z') = q(\lambda_1 x' + \lambda_2 y', \mu_1 x' + \mu_2 y', \nu_1 x' + \nu_2 y') + \lambda_3 z'\, q_1(\lambda_1 x' + \lambda_2 y', \mu_1 x' + \mu_2 y', \nu_1 x' + \nu_2 y') + \mu_3 z'\, q_2(\lambda_1 x' + \lambda_2 y', \mu_1 x' + \mu_2 y', \nu_1 x' + \nu_2 y') + \nu_3 z'\, q_3(\lambda_1 x' + \lambda_2 y', \ldots, \ldots) + q(\lambda_3 z', \mu_3 z', \nu_3 z').$$

To the first four terms on the right we apply again Taylor's theorem, remembering that $q_{ij} = 2\, a_{ij}$ (see Section 80); we find

then that

$$q'(x', y', z') = q(\lambda_1 x', \mu_1 x', \nu_1 x') + \lambda_2 y'\, q_1(\lambda_1 x', \mu_1 x', \nu_1 x') + \mu_2 y'\, q_2(\lambda_1 x', \mu_1 x', \nu_1 x') + \nu_2 y'\, q_3(\lambda_1 x', \mu_1 x', \nu_1 x') + q(\lambda_2 y', \mu_2 y', \nu_2 y') + \lambda_3 z' [q_1(\lambda_1 x', \mu_1 x', \nu_1 x') + 2\, a_{11}\lambda_2 y' + 2\, a_{12}\mu_2 y' + 2\, a_{13}\nu_2 y'] + \mu_3 z' [q_2(\lambda_1 x', \mu_1 x', \nu_1 x') + 2\, a_{21}\lambda_2 y' + 2\, a_{22}\mu_2 y' + 2\, a_{23}\nu_2 y'] + \nu_3 z' [q_3(\lambda_1 x', \mu_1 x', \nu_1 x') + 2\, a_{31}\lambda_2 y' + 2\, a_{32}\mu_2 y' + 2\, a_{33}\nu_2 y'] + q(\lambda_3 z', \mu_3 z', \nu_3 z').$$

We recall once more that q is a homogeneous function of the second degree and that q_1, q_2, and q_3 are homogeneous functions of the first degree; also the property of homogeneous functions of which we spoke in the proof of Theorem 3, Chapter VI (see Section 70, page 136). If we make use of these facts, we should be able to see that the second degree terms in $Q'(x', y', z')$ reduce, under a general rotation of axes, to

$$x'^2 q(\lambda_1, \mu_1, \nu_1) + y'^2 q(\lambda_2, \mu_2, \nu_2) + z'^2 q(\lambda_3, \mu_3, \nu_3) + x'y' [\lambda_2 q_1(\lambda_1, \mu_1, \nu_1) + \mu_2 q_2(\lambda_1, \mu_1, \nu_1) + \nu_2 q_3(\lambda_1, \mu_1, \nu_1)] + y'z' [\lambda_3 q_1(\lambda_2, \mu_2, \nu_2) + \mu_3 q_2(\lambda_2, \mu_2, \nu_2) + \nu_3 q_3(\lambda_2, \mu_2, \nu_2)] + z'x' [\lambda_3 q_1(\lambda_1, \mu_1, \nu_1) + \mu_3 q_2(\lambda_1, \mu_1, \nu_1) + \nu_3 q_3(\lambda_1, \mu_1, \nu_1)].$$

If in this expression we make use of the formulas established in the Corollary of Theorem 18, Chapter VII (see Section 88, page 190) for the direction cosines of the principal directions, this expression reduces to

$$k_1 x'^2 + k_2 y'^2 + k_3 z'^2 + 2\, k_1 x'y'(\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2) + 2\, k_2 y'z' (\lambda_2\lambda_3 + \mu_2\mu_3 + \nu_2\nu_3) + 2\, k_1 z'x'(\lambda_3\lambda_1 + \mu_3\mu_1 + \nu_3\nu_1).$$

Finally we put into operation the hypothesis that the new coördinate axes are mutually perpendicular and that therefore their direction cosines satisfy, two by two, the condition of Corollary 1 of Theorem 13, Chapter III (see Section 36, page 64); our second degree terms then become $k_1 x'^2 + k_2 y'^2 + k_3 z'^2$. We have therefore obtained the result stated in the following theorem.

THEOREM 10. **If the discriminating numbers of a quadric surface Q are k_1, k_2, and k_3, and if the frame of reference is rotated so that the new X-, Y-, and Z-axes have the directions of the principal directions determined by k_1, k_2, and k_3 respectively, then the equation of the surface with respect to the new frame is**

$$Q'(x', y', z') = k_1 x'^2 + k_2 y'^2 + k_3 z'^2 + q_4(\lambda_1, \mu_1, \nu_1)x' + q_4(\lambda_2, \mu_2, \nu_2)y' + q_4(\lambda_3, \mu_3, \nu_3)z' + a_{44} = 0.$$

Remark. The phrase "the principal directions determined by k_1, k_2, and k_3" used in the statement of this theorem is to be understood in the same sense as in Theorem 24, Chapter VII (see Section 90, page 195).

99. Classification of Quadric Surfaces — the Non-singular Cases. We are now in a position to analyze the general equation of the second degree in x, y, and z, that is, to determine the types of surfaces that can be represented by the equation $Q(x, y, z) = 0$. The problem of making this determination is usually referred to as the "classification of quadric surfaces."*

The analysis of the equation $Q(x, y, z) = 0$ will be based on the ranks r_4 and r_3 of the matrices $\mathbf{a}_4$ and $\mathbf{a}_3$ respectively; and we treat first those surfaces for which $r_4 = 4$, that is, the non-singular quadrics. In virtue of Corollary 1 of Theorem 14, Chapter VII (see Section 85, page 179), this condition carries with it that $r_3 \geqq 2$; we have therefore to consider two cases, namely, $r_4 = 4$, $r_3 = 3$; and $r_4 = 4$, $r_3 = 2$.

Case I. $r_4 = 4$, $r_3 = 3$.

We know from Theorem 14, Chapter VII (Section 85, page 178) that the surface has a single proper center. If the axes are translated to this center as an origin, the equation becomes (see Theorem 9, Section 97, page 211)

$$(1)\quad q(x', y', z') + \frac{\Delta}{A_{44}} = 0.$$

The three roots of the discriminating equation

$$(2)\quad k^3 - T_1k^2 + T_2k - A_{44} = 0$$

are all real and different from zero. Since the first degree terms are absent from equation (1), rotation of axes to principal directions will carry the equation over into

$$k_1x''^2 + k_2y''^2 + k_3z''^2 + \frac{\Delta}{A_{44}} = 0,$$

* This problem concerns itself therefore primarily with the question of determining what kind of surface is represented by given numerical equations and not with that of locating the position of the surface with respect to a frame of reference, nor with finding the particular numerical data which serve to specify the surface as an individual of its type. The method of treatment of our principal question is of such nature however as to develop means for answering these further questions.

k_1, k_2, k_3 being the roots of equation (2), (see Theorem 10, Section 98, page 213). Since $\Delta \neq 0$, this equation may be written in the form

$$\frac{x''^2}{-\dfrac{\Delta}{k_1A_{44}}} + \frac{y''^2}{-\dfrac{\Delta}{k_2A_{44}}} + \frac{z''^2}{-\dfrac{\Delta}{k_3A_{44}}} = 1.$$

This equation belongs to the types of equations whose loci were discussed in Section 72. If we make use of the discussion of this section, we reach the following conclusion:

(a) If $\dfrac{\Delta}{k_1A_{44}}, \dfrac{\Delta}{k_2A_{44}}, \dfrac{\Delta}{k_3A_{44}}$ are all negative, the surface is an **ellipsoid.**

(b) If two of these numbers are negative, the surface is an **hyperboloid of one sheet.**

(c) If one of these numbers is negative, the surface is an **hyperboloid of two sheets.**

(d) If none of these numbers is negative, the surface is an **imaginary ellipsoid.**

Remark. By reference to Example 2, Section 68, page 133, we see furthermore that, if the discriminating equation has a pair of equal roots, the quadric surface will be a surface of revolution, namely, an ellipsoid of revolution (real or imaginary), or a hyperboloid of revolution (of one sheet or of two sheets), according as the sign of the double root does or does not agree with that of the remaining root; if and only if the discriminating equation has a triple root, the quadric will be a sphere (real or imaginary).

We observe that the complete determination of the character of the surface depends in this case on the signs of Δ and A_{44}, and on the signs of the roots of the cubic equation (2), whose coefficients are all invariant with respect to translation and rotation of axes. Since the roots of this cubic are all real (compare Theorem 20, Chapter VII, Section 89, page 192), Descartes' Rule of Signs enables us to tell exactly how many positive and how many negative roots it has. If the signs of T_1 and of A_{44} are both changed, all the roots of the cubic change sign, and therefore the numbers $\dfrac{\Delta}{k_iA_{44}}$ preserve their signs for $i = 1, 2, 3$; hence we need consider

only the sign of the product T_1A_{44}.* We distinguish now the following cases:

(1) $\Delta > 0$, $T_2 > 0$. In accordance with the remark made above, the sequences of sign in the cubic which have to be considered are the following:

When $A_{44} > 0$, $T_1 > 0$, the signs are $+ - + -$;
and, when $A_{44} > 0$, $T_1 < 0$, the signs are $+ + + -$.

If the first of these occurs, the equation has three positive roots and the three "coefficients" $\dfrac{\Delta}{k_iA_{44}}$, $i = 1, 2, 3$ are positive; the surface is therefore an imaginary ellipsoid; if the second sequence occurs, there are two negative roots and one positive root and hence two negative and one positive coefficients, so that the surface is an hyperboloid of one sheet.

(2) $\Delta < 0$, $T_2 > 0$. The sequences of sign are the same as before, but since now $\Delta < 0$, all the coefficients will have changed their signs. Therefore we shall have an ellipsoid if $A_{44}T_1 > 0$, and an hyperboloid of two sheets if $A_{44}T_1 < 0$.

(3) $\Delta > 0$, $T_2 < 0$. If $A_{44} > 0$, the sequence of signs will be $+ + - -$ or $+ - - -$, so that we have one positive and two negative roots and also one positive and two negative coefficients. If $A_{44} < 0$, the sequences of signs are $+ + - +$ or $+ - - +$, so that there are one negative and two positive roots but, since A_{44} has changed sign, again one positive and two negative coefficients. In this case therefore the surface is always an hyperboloid of one sheet.

(4) $\Delta < 0$, $T_2 < 0$. We have the same distribution of roots as in (3), but, since Δ has the opposite sign, the coefficients will be opposite in sign; the surface is therefore an hyperboloid of two sheets.

It remains to consider the cases in which either T_1 or T_2 vanishes; that they can not vanish simultaneously was shown in the Corollary of Theorem 21, Chapter VII (see Section 89, page 194). If either

* It should be clear that the signs of T_1 and of A_{44} can not be significant in determining the character of the locus of the equation $Q = 0$. For, if this equation is multiplied through by -1, T_1 and A_{44} clearly change their signs, but the locus of the equation is obviously not affected. This remark does not apply to T_2, Δ or T_1A_{44}.

T_1 or T_2 vanishes, the roots can not all have the same sign; for since $T_1 = k_1 + k_2 + k_3$ and $T_2 = k_1k_2 + k_2k_3 + k_3k_1$, the former of these expressions would then have the sign common to the roots and the latter would be positive. Moreover $A_{44} = k_1k_2k_3$; hence, if $\Delta > 0$ and $A_{44} > 0$, there must be one positive and two negative roots and also one positive and two negative coefficients, and if $\Delta > 0$ and $A_{44} < 0$, there are one negative and two positive roots, but again one positive and two negative coefficients. In either case the surface is an hyperboloid of one sheet.

But if $\Delta < 0$, there will be one negative and two positive coefficients, so that the surface is an hyperboloid of two sheets.

We summarize the results in the following theorem.

THEOREM 11. **If the ranks of the matrices a_4 and a_3 are 4 and 3 respectively, the locus of the equation $Q = 0$ will be determined by the following table:**

	$\Delta > 0$	$\Delta < 0$
$T_2 > 0, \quad A_{44}T_1 > 0$	**Imaginary ellipsoid**	**Ellipsoid**
$T_2 > 0, \quad A_{44}T_1 \leqq 0$ or $T_2 \leqq 0$	**Hyperboloid of one sheet**	**Hyperboloid of two sheets**

Remark. We observe that the character of the surface can be completely determined in this case as soon as the signs of the invariants Δ, A_{44}, T_2 and T_1 are known and that it is not necessary for this purpose to solve the discriminating equation. Compare also the Remark on page 215.

CASE II. $r_4 = 4, \ r_3 = 2.$

According to Theorem 14, Chapter VII, the surface does not have a center in this case. Since A_{44} is equal to zero, T_2 must be different from zero, for otherwise we could conclude by means of the Corollary of Theorem 7, Chapter II (see Section 26, page 44) that $r_3 < 2$. Consequently, one and only one root of the equation (2) vanishes; let it be k_1. Rotation of axes to principal directions will then reduce the equation $Q = 0$ to the form

$$(3) \quad k_2y'^2 + k_3z'^2 + q_4(\lambda_1, \mu_1, \nu_1)x' + q_4(\lambda_2, \mu_2, \nu_2)y' + q_4(\lambda_3, \mu_3, \nu_3)z' + a_{44} = 0.$$

The discriminant of this equation is

$$\begin{vmatrix} 0 & 0 & 0 & a_{14}' \\ 0 & k_2 & 0 & a_{24}' \\ 0 & 0 & k_3 & a_{34}' \\ a_{14}' & a_{24}' & a_{34}' & a_{44} \end{vmatrix} = -a_{14}'^2 k_2 k_3,$$

where $2\,a_{14}'$, $2\,a_{24}'$ and $2\,a_{34}'$ are used, as before, to designate the coefficients $q_4(\lambda_1, \mu_1, \nu_1)$, $q_4(\lambda_2, \mu_2, \nu_2)$ and $q_4(\lambda_3, \mu_3, \nu_3)$ of x', y', and z' respectively. But the discriminant of a quadric surface is invariant with respect to rotation of axes (see Theorem 4, Section 94, page 203); hence $\Delta = -a_{14}'^2 k_2 k_3$ and, since $k_2 \neq 0$ and $k_3 \neq 0$, $a_{14}' = \pm\sqrt{-\dfrac{\Delta}{k_2 k_3}} \neq 0$. The further reduction of the equation is now made as follows; completing the square on the terms in y' and z', it becomes

$$k_2\left(y' + \frac{a_{24}'}{k_2}\right)^2 + k_3\left(z' + \frac{a_{34}'}{k_3}\right)^2$$
$$= -2\,a_{14}'\left(x' + \frac{a_{44}}{2\,a_{14}'} - \frac{a_{24}'^2}{2\,a_{14}'k_2} - \frac{a_{34}'^2}{2\,a_{14}'k_3}\right).$$

We translate the axes now to the point

$$\left(-\frac{a_{44}}{2\,a_{14}'} + \frac{a_{24}'^2}{2\,a_{14}'k_2} + \frac{a_{34}'^2}{2\,a_{14}'k_3},\ -\frac{a_{24}'}{k_2},\ -\frac{a_{34}'}{k_3}\right)$$

as origin by putting

$$x' = x'' - \frac{a_{44}}{2\,a_{14}'} + \frac{a_{24}'^2}{2\,a_{14}'k_2} + \frac{a_{34}'^2}{2\,a_{14}'k_3},\quad y' = y'' - \frac{a_{24}'}{k_2},$$
$$z' = z'' - \frac{a_{34}'}{k_3}.$$

This transformation carries the equation of the surface over into the form

$$k_2 y''^2 + k_3 z''^2 = -2\,a_{14}'x''.$$

We reach therefore the conclusion, by means of the results of Section 72, that in this case the locus of the equation $Q = 0$ is an **elliptic paraboloid** if k_2 and k_3 have the same sign, and an **hyperbolic paraboloid** if they are opposite in sign. But k_1 and k_2 are the roots of the quadratic equation $k^2 - T_1 k + T_2 = 0$, and therefore the first or the second of these cases will arise according as

T_2 is positive or negative. We shall replace this criterion by another one; but before doing so we observe that the surface will be a paraboloid of revolution if and only if $k_2 = k_3$.

We will prove now the following theorem.

THEOREM 12. **If $A_{44} = 0$, then Δ is the square of a linear homogeneous function of a_{14}, a_{24}, and a_{34}.**

Proof. Since $A_{44} = 0$, the development of Δ according to the elements of its last column leads to the equation

$$\Delta = -a_{14}\begin{vmatrix} a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{vmatrix} + a_{24}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{vmatrix} - a_{34}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{14} & a_{24} & a_{34} \end{vmatrix}.$$

If we develop each of these three-rowed determinants according to the elements of their last row and use the notation α_{ij} for the cofactors of the elements a_{ij} of the matrix $\mathbf{a}_3$, as introduced on page 184 in the proof of Theorem 16, Chapter VII, we find that

$$\begin{aligned} \Delta = & -a_{14}(a_{14}\alpha_{11} + a_{24}\alpha_{12} + a_{34}\alpha_{13}) + a_{24}(-a_{14}\alpha_{12} - a_{24}\alpha_{22} - a_{34}\alpha_{23}) \\ & - a_{34}(a_{14}\alpha_{13} + a_{24}\alpha_{23} + a_{34}\alpha_{33}) \\ = & -(a_{14}{}^2\alpha_{11} + 2\, a_{14}a_{24}\alpha_{12} + 2\, a_{14}a_{34}\alpha_{13} + a_{24}{}^2\alpha_{22} + 2\, a_{24}a_{34}\alpha_{23} \\ & + a_{34}{}^2\alpha_{33}). \end{aligned}$$

It follows from Corollary 2 of Theorem 19, Chapter VII (Section 89, page 192), since we are supposing that $A_{44} = 0$, that $\alpha_{ii}\alpha_{jj} = \alpha_{ij}{}^2$, for $i, j = 1, 2, 3$; therefore $\alpha_{ij} = \pm\sqrt{\alpha_{ii}\alpha_{jj}}$, so that we may write

$$\Delta = -(a_{14}\sqrt{\alpha_{11}} \pm a_{24}\sqrt{\alpha_{22}} \pm a_{34}\sqrt{\alpha_{33}})^2$$

which proves our theorem, since the negative sign outside the parentheses may be introduced under each of the radicals.

From this theorem we derive an important corollary. For the discussion recalls, as might also be derived from Corollary 3 of Theorem 19, Chapter VII (Section 89, page 192), that those of the principal minors α_{11}, α_{22}, α_{33} which do not vanish have the same sign as T_2 (and not all of them can vanish if $r_3 = 2$). Hence, if $T_2 > 0$, $\sqrt{\alpha_{11}}$, $\sqrt{\alpha_{22}}$, and $\sqrt{\alpha_{33}}$ are real and $\Delta \leqq 0$; while if $T_2 < 0$, these square roots are pure imaginaries or zero (not all zero) and $\Delta > 0$.

COROLLARY 1. **If $A_{44} = 0$, $T_2 \neq 0$ and $\Delta \neq 0$, then T_2 and Δ are opposite in sign.**

The method used to prove Theorem 12 enables us also to prove the following important formula:

Corollary 2. **The value of the determinant** $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a \\ a_{12} & a_{22} & a_{23} & b \\ a_{13} & a_{23} & a_{33} & c \\ a & b & c & 0 \end{vmatrix}$ **is equal to** $-(\alpha_{11}a^2 + \alpha_{22}b^2 + \alpha_{33}c^2 + 2\,\alpha_{23}bc + 2\,\alpha_{13}ca + 2\,\alpha_{12}ab)$.

Returning now to the discussion which precedes Theorem 12, we can state the following theorem.

Theorem 13. **If the rank of the matrices a_4 and a_3 are 4 and 2 respectively, the locus of the equation $Q = 0$ is an elliptic paraboloid if $\Delta < 0$, and an hyperbolic paraboloid if $\Delta > 0$.**

100. Classification of Quadric Surfaces — the Non-degenerate Singular Cases. If $r_4 = 3$, we can have $r_3 = 3$, 2 or 1.

Case III. $r_4 = 3,\ r_3 = 3$.

From Theorem 14, Chapter VII, we know that in this case the surface has a single vertex and from Corollary 3 of this theorem we know that it is a proper quadric cone. The reduction of the equation in this case is made in exactly the same way as in Case I, except that we have now $\dfrac{\Delta}{A_{44}} = 0$, so that the final form of the equation is

$$k_1x''^2 + k_2y''^2 + k_3z''^2 = 0.$$

The cone is real if and only if the discriminating numbers do not all have the same sign; this will always be the case unless the coefficients in the discriminating equation present either no variations or three variations of sign, that is, unless $T_2 > 0$ and $A_{44}T_1 > 0$. In this case we have therefore the following result:

Theorem 14. **If the ranks of the matrices a_4 and a_3 are both equal to 3, the locus of the equation $Q = 0$ is an imaginary cone, if $T_2 > 0$ and $A_{44}T_1 > 0$; in all other cases the locus will be a real quadric cone.**

Remark. The surface will be a real circular cone if and only if the discriminating equation has a simple root of one sign and a double root of the opposite sign.

From Theorem 14 we shall derive an important algebraic theorem. It is an immediate consequence of Theorem 14 that the equation

$$q(x, y, z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2\,a_{23}yz + 2\,a_{31}zx + 2\,a_{12}xy = 0$$

represents a cone if the determinant $A_{44} = |a_{ij}|$, $i, j = 1, 2, 3$ is different from zero; this cone will be imaginary if $T_2 = \alpha_{11} + \alpha_{22} + \alpha_{33} > 0$ and $A_{44}T_1 = A_{44}(a_{11} + a_{22} + a_{33}) > 0$, but in all other cases it is real.

In the first case, the function $q(x, y, z)$ is reducible to the form $k_1x^2 + k_2y^2 + k_3z^2$, in which k_1, k_2, and k_3 are different from zero and are of like sign; the function $q(x, y, z)$ will be zero if $x = y = z = 0$ and it will be of one sign for all other sets of real values of the variables, namely, of the sign of its coefficients, which will be the sign of A_{44} since A_{44} is equal to $k_1k_2k_3$. In the second case the function $q(x, y, z)$ is also reducible to the form $k_1x^2 + k_2y^2 + k_3z^2$, but now the coefficients in this form are not all of the same sign, and the function can therefore take negative, positive and zero values for different sets of real values of the variables. We introduce now the following definitions.

Definition II. **A homogeneous function of degree 2 in 3 variables is called a *quadratic ternary form*.***

Definition III. **A *positive* (*negative*) *definite form* is one which takes the value zero only when all the variables vanish and is positive (negative) for all other sets of real values of the variables; an indefinite form is one which can take positive, negative and zero values for real values of the variables.**

We can now state the following important algebraic theorem.

Theorem 15. **The quadratic ternary form $q(x, y, z)$ for which the determinant A_{44} does not vanish is definite if and only if $\alpha_{11} + \alpha_{22} + \alpha_{33} > 0$ and $A_{44}(a_{11} + a_{22} + a_{33}) > 0$; it is positive or negative definite according as A_{44} is positive or negative.**

Case IV. $r_4 = 3, r_3 = 2$.

It follows from Theorem 14, Chapter VII, that the surface has a line of centers. We could therefore begin by translating axes to one of the centers as origin; but the reduction of the equation is accomplished more rapidly if we follow the method used in Case II. Rotation of axes to principal directions leads again to equation (3) of Section 99 (see page 217); but since now $\Delta = 0$ and since the discriminant is invariant under rotation, we conclude from the discussion made in Case II (page 218) that $a_{14}' = 0$. The equation of the surface reduces therefore to the form

$$(1) \quad k_2y'^2 + k_3z'^2 + 2\,a_{24}'y' + 2\,a_{34}'z' + a_{44} = 0.$$

* A homogeneous polynomial of degree 3, 4, . . . , n is called a cubic, quartic, . . . , n-ic form; a form in 2 variables is called a binary form, one in 4, 5, . . . , n variables is called quaternary, quinary, . . . , n-ary.

Completing the square on the terms in y' and on the terms in z' and translating the origin to an arbitrary point on the line $y' = -\dfrac{a_{24}'}{k_2}$, $z' = -\dfrac{a_{34}'}{k_3}$ leads to the equation

$$k_2 y''^2 + k_3 z''^2 = a_{44}'',$$

where

$$y'' = y' + \frac{a_{24}'}{k_2}, \quad z'' = z' + \frac{a_{34}'}{k_3} \text{ and } a_{44}'' = \frac{a_{24}'^2}{k_2} + \frac{a_{34}'^2}{k_3} - a_{44}.$$

The discriminant of this last equation is

$$\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & -a_{44}'' \end{vmatrix}.$$

It will clearly be of rank 2, unless $a_{44}'' \neq 0$. But, since $r_4 = 3$ and the rank of the discriminant matrix is invariant under rotation and translation of axes (compare Corollary 3 of Theorem 1, Section 93, page 201 and Theorem 5, Section 94, page 205), we conclude that $a_{44}'' \neq 0$. The locus of the equation is therefore a cylindrical surface; it will be an hyperbolic cylinder if k_2 and k_3 are opposite in sign, a real elliptic cylinder if k_2, k_3, and a_{44}'' are of like sign, an imaginary cylinder if k_2 and k_3 are of like sign, opposite to that of a_{44}''.

As in Case II, we see that $k_2 k_3 = T_2$, so that k_2 and k_3 will have the same sign or opposite signs according as $T_2 > 0$ or $T_2 < 0$; in the former case, they will have the sign of $T_1 = k_2 + k_3$. To determine whether or not, in case k_2 and k_3 are of like sign, their sign is the same as that of a_{44}'', we consider the sum D_3 of the three-rowed minors of the discriminant. Since equation (1) was obtained from the original equation $Q = 0$ by rotation of axes, we know from the Corollary of Theorem 4 (see Section 94, page 204) that $D_3' = D_3$. The discriminant of equation (1) is

$$\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & a_{24}' \\ 0 & 0 & k_3 & a_{34}' \\ 0 & a_{24}' & a_{34}' & a_{44} \end{vmatrix}$$

and we see that every three-rowed minor of the matrix **b**, associated with this discriminant, vanishes. We conclude therefore, by making use of the theorem stated in Exercise 6, Section 95 (page

205), that $D_3'' = D_3' = D_3$. Now $D_3'' = -a_{44}''k_2k_3$; therefore $a_{44}''k_2k_3 = -D_3$. This relation enables us to say that if $T_2 = k_2k_3 > 0$, a_{44}'' will be opposite in sign to D_3. Since moreover the signs of k_2 and k_3 are the same as that of T_1, we conclude that k_2, k_3, and a_{44}'' will have one sign if T_1D_3 is negative, but the sign of a_{44}'' will be opposite to that of k_2 and k_3 if T_1D_3 is positive. We have therefore reached the following conclusion.

THEOREM 16. **If the ranks of the matrices $\mathbf{a}_4$ and $\mathbf{a}_3$ are equal to 3 and 2 respectively, the locus of the equation $Q = 0$ will be a real elliptic cylinder if and only if $T_2 > 0$ and $T_1D_3 < 0$, an imaginary cylinder if and only if $T_2 > 0$ and $T_1D_3 > 0$, an hyperbolic cylinder if and only if $T_2 < 0$.**

Remark 1. We observe that, as in Case II, T_2 must be different from zero in this case.

Remark 2. The surface will be a circular cylinder if and only if $k_2 = k_3$.

CASE V. $r_4 = 3$, $r_3 = 1$.

The surface has no center in this case. Both T_2 and A_{44} are equal to zero, but T_1 is different from zero; for otherwise it would follow that $r_3 = 0$ by means of an argument which is entirely similar to the argument in earlier discussions and which is therefore left to the reader. The discriminating equation is now

$$k^3 - T_1k^2 = 0.$$

Its roots are $k_1 = k_2 = 0$ and $k_3 = T_1 \neq 0$. In accordance with Theorem 24, Chapter VII (Section 90, page 195), only one principal direction is completely determined, namely, λ_3, μ_3, ν_3; the other two principal directions are subject only to the condition of perpendicularity to this first direction, and to mutual perpendicularity. We are therefore free to impose one additional condition on λ_1, μ_1, ν_1 or on λ_2, μ_2, ν_2.

It is easy to show that in this case $\lambda_3 : \mu_3 : \nu_3 = a_{i1} : a_{i2} : a_{i3}$, $i = 1, 2, 3$. For, since $k_3 = T_1 = a_{11} + a_{22} + a_{33}$, we can determine λ_3, μ_3, and ν_3 from any two of the three linear equations

$$-(a_{22} + a_{33})\lambda_3 + a_{12}\mu_3 + a_{13}\nu_3 = 0, \quad a_{12}\lambda_3 - (a_{11} + a_{33})\mu_3 + a_{23}\nu_3 = 0, \quad a_{13}\lambda_3 + a_{23}\mu_3 - (a_{11} + a_{22})\nu_3 = 0.$$

From the first two of these equations we find

$$\lambda_3 : \mu_3 : \nu_3 = a_{12}a_{23} + a_{13}(a_{11} + a_{33}) : a_{13}a_{12} + a_{23}(a_{22} + a_{33}) : (a_{11} + a_{33})(a_{22} + a_{33}) - a_{12}^2.$$

But $r_3 = 1$; hence $\alpha_{13} = a_{12}a_{23} - a_{13}a_{22} = 0$, so that $a_{12}a_{23} = a_{13}a_{22}$.
Also $\alpha_{23} = a_{13}a_{12} - a_{11}a_{23} = 0$, so that $a_{13}a_{12} = a_{11}a_{23}$.
And $\alpha_{33} = a_{11}a_{22} - a_{12}^2 = 0$.

Consequently we find that $\lambda_3 : \mu_3 : \nu_3 = a_{13}T_1 : a_{23}T_1 : a_{33}T_1$. And since $T_1 \neq 0$ and $r_3 = 1$, so that the rows of the matrix $\mathbf{a}_3$ are proportional, we reach the conclusion that $\lambda_3 : \mu_3 : \nu_3 = a_{i1} : a_{i2} : a_{i3}$, $i = 1, 2, 3$.

If we rotate axes to the principal directions determined in accordance with these methods, the equation $Q = 0$ will be carried over to the form

$$k_3z'^2 + 2\,a_{14}'x' + 2\,a_{24}'y' + 2\,a_{34}'z' + a_{44} = 0,$$

where, as before, $2\,a_{i4}' = q_4(\lambda_i, \mu_i, \nu_i)$, $i = 1, 2, 3$. The matrix $\mathbf{a}_4'$ of this reduced equation is

$$\begin{Vmatrix} 0 & 0 & 0 & a_{14}' \\ 0 & 0 & 0 & a_{24}' \\ 0 & 0 & k_3 & a_{34}' \\ a_{14}' & a_{24}' & a_{34}' & a_{44} \end{Vmatrix}.$$

In virtue of the hypothesis $r_4 = 3$ and of Theorem 5 (Section 94, page 205), the rank of this matrix must be 3; since the matrix is obviously singular, it must contain, on the basis of the Corollary of Theorem 6, Chapter II (Section 26, page 44), at least one non-vanishing three-rowed principal minor. It should be easy to see that the only three-rowed principal minors of this matrix which do not vanish identically are those formed from 1st, 3rd, and 4th rows and columns, and from the 2nd, 3rd, and 4th rows and columns; also that the values of these are $-k_3a_{14}'^2$ and $-k_3a_{24}'^2$. It follows that at least one of the numbers a_{14}' and a_{24}' must be different from zero. And now we make use of the freedom of choice left in the determination of either λ_1, μ_1, ν_1 or λ_2, μ_2, ν_2 to effect a further simplification of the equation.

If we adjoin the condition $q_4(\lambda_1, \mu_1, \nu_1) = 0$ to the condition $\lambda_1\lambda_3 + \mu_1\mu_3 + \nu_1\nu_3 = 0$, which is imposed by the condition of perpendicularity of the principal directions, the direction λ_1, μ_1, ν_1 is determined; and then λ_2, μ_2, ν_2 will also be determined as the direction perpendicular to the other two. In this manner we secure the result that $a_{14}' = 0$ and therefore also the fact that $a_{24}' \neq 0$. Since $\lambda_3 : \mu_3 : \nu_3 = a_{i1} : a_{i2} : a_{i3}$, $i = 1, 2, 3$, we can determine

λ_1, μ_1, ν_1 from any one of the 3 systems of two equations each given by $a_{14}\lambda_1 + a_{24}\mu_1 + a_{34}\nu_1 = 0$ together with one of the equations $a_{i1}\lambda_1 + a_{i2}\mu_1 + a_{i3}\nu_1 = 0$. Hence λ_1, μ_1, and ν_1 are proportional to the two-rowed determinants formed from one of the three matrices $\begin{Vmatrix} a_{i1} & a_{i2} & a_{i3} \\ a_{14} & a_{24} & a_{34} \end{Vmatrix}$, $i = 1, 2, 3$. This will always determine these direction cosines, unless every two-rowed minor of the matrix **b** vanished; but in this case the rank of the matrix **b** would be 1 and this is incompatible with the condition $r_4 = 3$, in view of the observation (3) made in the proof of Theorem 14, Chapter VII (see page 178). We conclude therefore that the principal directions can in this case be so determined that $a_{14}' = 0$ and $a_{24}' \neq 0$. The equation of the surface thus takes the form

$$k_3 z'^2 + 2\, a_{24}' y' + 2\, a_{34}' z' + a_{44} = 0.$$

If we complete the square on the terms in z, this equation finally reduces to

$$k_3 z''^2 = -2\, a_{24}' y''$$

where $x'' = x'$, $\quad y'' = y' + \dfrac{a_{44}}{2\, a_{24}'} - \dfrac{a_{34}'^2}{2\, a_{24}' k_3}$, $\quad z'' = z' + \dfrac{a_{34}'}{k_3}$.

The locus of this equation is a parabolic cylinder. We may therefore state the following conclusion.

THEOREM 17. **If the ranks of the matrices $\mathbf{a}_4$ and $\mathbf{a}_3$ are 3 and 1 respectively, the locus of the equation $Q = 0$ is a parabolic cylinder.**

Remark 1. It should be obvious that we might equally well have determined the direction cosines λ_2, μ_2, ν_2 in such a way that $a_{24}' = 0$ and $a_{14}' \neq 0$. In this case the final equation would become $k_3 z''^2 = -2\, a_{14}' x''$, whose locus is also a parabolic cylinder. Indeed this change merely amounts to an interchange of the X''- and Y''-axes.

Remark 2. It follows from Corollary 3 of Theorem 8 (Section 96, page 209) that in the case just treated the function $q(x, y, z)$ is the square of a linear homogeneous function of x, y, z with real or complex coefficients. This observation is frequently useful for recognizing whether or not the equation $Q = 0$ represents a parabolic cylinder.

Example.

To analyze the equation

$$4\,x^2 + y^2 + 4\,z^2 - 4\,xy - 4\,yz + 8\,zx + 2\,x - 4\,y + 3\,z + 1 = 0,$$

we set up the matrices $\mathbf{a}_4 = \left\|\begin{matrix} 4 & -2 & 4 & 1 \\ -2 & 1 & -2 & -2 \\ 4 & -2 & 4 & \frac{3}{2} \\ 1 & -2 & \frac{3}{2} & 1 \end{matrix}\right\|$ and $\mathbf{a}_3 = \left\|\begin{matrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{matrix}\right\|$.

It is obvious that $r_3 = 1$; hence $r_4 < 4$; and since the three-rowed minor in the lower right-hand corner of $\mathbf{a}_4$, namely, the determinant $\begin{vmatrix} 1 & -2 & -2 \\ -2 & 4 & \frac{3}{2} \\ -2 & \frac{3}{2} & 1 \end{vmatrix}$ has the value $-\frac{25}{4}$, $r_3 = 3$.

In accordance with Theorem 17, we conclude therefore that the locus of the equation is a parabolic cylinder. This settles the question as to the type of surface represented by the equation. We proceed now to determine its position with reference to the given system of coördinates, partly in order to exemplify and to verify the method used in the discussion of Case V, and partly in illustration of the remark made in the footnote on page 214.

From the matrix $\mathbf{a}_3$ we conclude furthermore that $T_1 = 9$ and we verify that $T_2 = 0$. The discriminating equation is therefore $k^3 - 9k^2 = 0$, so that we may take $k_1 = k_2 = 0$ and $k_3 = 9$. To determine λ_3, μ_3, and ν_3 we have the equations

$$-5\lambda_3 - 2\mu_3 + 4\nu_3 = 0, \quad -2\lambda_3 - 8\mu_3 - 2\nu_3 = 0, \quad 4\lambda_3 - 2\mu_3 - 5\nu_3 = 0.$$

From any two of these three equations we obtain $\lambda_3 : \mu_3 : \nu_3 = 2 : -1 : 2$, a result which was predictable from the discussion in the first part of Case V. Since $k_1 = k_2 = 0$, the conditions for λ_1, μ_1, ν_1 (and also those for λ_2, μ_2, ν_2) reduce to the single equation $2\lambda_1 - \mu_1 + 2\nu_1 = 0$, which expresses the condition of perpendicularity to the direction λ_3, μ_3, ν_3. To this condition we adjoin the condition $q_4(\lambda_1, \mu_1, \nu_1) = 2\lambda_1 - 4\mu_1 + 3\nu_1 = 0$. From these two conditions we find then that $\lambda_1 : \mu_1 : \nu_1 = 5 : -2 : -6$. For λ_2, μ_2, ν_2 we have now the conditions

$$2\lambda_2 - \mu_2 + 2\nu_2 = 0 \quad \text{and} \quad 5\lambda_2 - 2\mu_2 - 6\nu_2 = 0$$

which express the condition of perpendicularity to the two directions already determined; from them we find that $\lambda_2 : \mu_2 : \nu_2 = 10 : 22 : 1$.

The rotation of axes to principal directions is therefore based on the following table (compare Section 63):

	X	Y	Z
X'	$\frac{5}{\sqrt{65}}$	$\frac{-2}{\sqrt{65}}$	$\frac{-6}{\sqrt{65}}$
Y'	$\frac{10}{3\sqrt{65}}$	$\frac{22}{3\sqrt{65}}$	$\frac{1}{3\sqrt{65}}$
Z'	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$

The equations of transformation are therefore

$$x = \frac{5\,x'}{\sqrt{65}} + \frac{10\,y'}{3\sqrt{65}} + \frac{2\,z'}{3}, \quad y = -\frac{2\,x'}{\sqrt{65}} + \frac{22\,y'}{3\sqrt{65}} - \frac{z'}{3}, \quad z = -\frac{6\,x'}{\sqrt{65}} + \frac{y'}{3\sqrt{65}} + \frac{2\,z'}{3}.$$

Hence*

$$Q(x, y, z) = (2\,x - y + 2\,z)^2 + 2\,x - 4\,y + 3\,z + 1 = (3\,z')^2 - \frac{\sqrt{65}\,y'}{3} + \frac{14\,z'}{3} + 1.$$

The equation of the given surface with respect to the rotated axes may therefore be successively transformed as follows:

$$\left(3\,z' + \frac{7}{9}\right)^2 = \frac{\sqrt{65}\,y'}{3} - \frac{32}{81},$$

$$9\left(z' + \frac{7}{27}\right)^2 = \frac{\sqrt{65}}{3}\cdot\left(y' - 32\cdot\frac{3}{81\sqrt{65}}\right),$$

$$z''^2 = \frac{\sqrt{65}\,y''}{27}.$$

This is the equation of the parabolic cylinder with respect to a frame of reference obtained from the original frame by first rotating the axes in accordance with the table indicated above and then translating the rotated axes to a new origin whose coördinates with respect to the rotated axes are $x' = 0$, $y' = 32\cdot\frac{3}{81\sqrt{65}}$, $z' = -\frac{7}{27}$. The point determined by these coördinates is the vertex of a directrix parabola on the cylinder.

101. Classification of Quadric Surfaces — the Degenerate Cases. There remain to be considered the cases $r_4 = 2$, $r_3 = 2$; $r_4 = 2$, $r_3 = 1$, and $r_4 = r_3 = 1$. These cases have already been discussed in Section 96 (see Theorem 7, page 208 and Theorem 8, page 209) and the results stated there completely settle the problem of classification for this case. It will however be instructive to derive these results also by means of the methods of reduction which were used in Sections 99 and 100.

Case VI. $r_4 = r_3 = 2$.

The discriminating equation has the form $k^3 - T_1k^2 + T_2k = 0$, where $T_2 \neq 0$. As in Case IV, rotation of axes to principal directions leads the equation $Q = 0$ over into the equation

$$k_2y'^2 + k_3z'^2 + 2\,a_{24}'y' + 2\,a_{34}'z' + a_{44} = 0.$$

* Compare Remark 2 following Theorem 17, page 225.

Completing the square on the terms in y' and z' and translating axes, we obtain the equation

$$k_2y''^2 + k_3z''^2 = a_{44}'',$$

where $a_{44}'' = \dfrac{a_{24}'^2}{k_2} + \dfrac{a_{34}'^2}{k_3} - a_{44}$, and the new origin is any point on the line $y' = -\dfrac{a_{24}'}{k_2}$, $z' = -\dfrac{a_{34}'}{k_3}$.

It should now be easy to show that, since $r_4 = 2$ and since the rank of the discriminant is invariant under rotation and translation of axes, $a_{44}'' = 0$. The final equation is therefore

$$k_2y''^2 + k_3z''^2 = 0;$$

and this equation represents a pair of intersecting planes, real if $T_2 = k_2k_3$ is negative, imaginary if T_2 is positive; this is the result stated in Theorem 7, page 208.

Case VII. $r_4 = 2$, $r_3 = 1$.

The discriminating equation has the same form as in Case V and rotation of axes to principal directions leads again to the equation

$$k_3z'^2 + 2\,a_{14}'x' + 2\,a_{24}'y' + 2\,a_{34}'z' + a_{44} = 0.$$

The argument used in the discussion of Case V (see page 224) shows that, since now $r_4 = 2$, $a_{24}' = a_{34}' = 0$. Completing the square and translating the axes reduces this equation to the form

$$k_3z''^2 + a_{44}'' = 0,$$

where $a_{44}'' = a_{44} - \dfrac{a_{34}'^2}{k_3}$ and where the new origin is any point on the plane $z' = -\dfrac{a_{34}'}{k_3}$. The sum D_2'' of the two-rowed principal minors of the discriminant of this last equation is clearly equal to k_3a_{44}'' and this is the only two-rowed minor of the discriminant which does not vanish identically. Since $r_4 = 2$, this can not vanish and therefore $a_{44}'' \neq 0$. Moreover, an argument similar to the one used in the discussion of Case IV (see page 222) shows that the sum D_2'' for the final reduced equation is the same as the sum D_2 for the original equation $Q = 0$. (The details of this argument are left to the reader.) Therefore $k_3a_{44}'' = D_2$, so that k_3 and a_{44}'' will be of like or of unlike signs according as D_2 is positive or negative. We conclude therefore that the locus of the equation $Q = 0$

is, in this case, a pair of parallel planes which are real or imaginary, according as D_2 is negative or positive; this result is stated in the first part of Theorem 8 (see page 209).

CASE VIII. $r_4 = 1, \quad r_3 = 1.$

In this case rotation of axes to principal directions and translation of axes, as in Case VII, leads to the final equation

$$k_3 z''^2 = 0,$$

which represents a pair of coincident planes, in accord with the last part of Theorem 8.

From this discussion we derive some further algebraic theorems, which complement Theorem 15 (see page 221).

If the rank of the matrix $\mathbf{a}_3$ is 2, the equation $q(x, y, z) = 0$ represents a pair of intersecting planes, which are real or imaginary according as T_2 is negative or positive. In the latter case the function $q(x, y, z)$ is reducible to the form $k_1x^2 + k_2y^2$, in which k_1 and k_2 have like sign, namely, the sign of T_1, which is equal to $k_1 + k_2$; the function is therefore a definite quadratic ternary form, positive definite or negative definite according as $T_1 > 0$ or < 0. If T_2 is negative, the function $q(x, y, z)$ is reducible to $k_1x^2 + k_2y^2$, and k_1 and k_2 will be opposite in sign; in this case the function is an indefinite form.

If the rank of the matrix $\mathbf{a}_3$ is 1, the equation $q(x, y, z) = 0$ represents a pair of coincident planes; the function $q(x, y, z)$ is therefore reducible to the form T_1x^2, which is a definite form, positive or negative, according as $T_1 > 0$ or < 0.

We have therefore the following extension of Theorem 15.

THEOREM 18. **The quadratic ternary form $q(x, y, z)$ for which the rank of the matrix $\mathbf{a}_3$ is 2 is definite if and only if $T_2 > 0$, positive definite or negative definite, according as T_1 is positive or negative; if the rank of the matrix $\mathbf{a}_3$ is 1, $q(x, y, z)$ is a definite form, positive definite or negative definite according as T_1 is positive or negative.**

102. The Classification of Quadric Surfaces — Summary and Geometric Characterization. The results which have been obtained in Sections 99, 100, 101, in as far as they relate to the classification of quadric surfaces, are summarized in the following table, which specifies the type of surface represented by the equation $Q(x, y, z) = 0$ in terms of the invariants of this equation. We indicate once more the meaning of each of the symbols used in the table.

$$\Delta = |a_{ij}|,\ i, j = 1, 2, 3, 4;$$
$$A_{44} = |a_{ij}|,\ i, j = 1, 2, 3;$$
$$D_1 = \sum_{i=1}^{4} a_{ii};\quad D_2 = \sum_{\substack{i,j=1\\ i<j}}^{4} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ij} & a_{jj} \end{vmatrix};\quad D_3 = \sum_{\substack{i,j,k=1\\ i<j<k}}^{4} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ij} & a_{jj} & a_{jk} \\ a_{ik} & a_{jk} & a_{kk} \end{vmatrix};$$
$$T_1 = \sum_{i=1}^{3} a_{ii};\quad T_2 = \sum_{\substack{i,j=1\\ i<j}}^{3} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ij} & a_{jj} \end{vmatrix};$$

r_4 = rank of matrix of Δ; r_3 = rank of matrix of A_{44}.

<table>
<tr><td>$r_3 \backslash r_4$</td><td colspan="3">4
Non-singular quadrics</td><td colspan="2">3
Singular non-degenerate quadrics</td><td colspan="2">2
Degenerate quadrics</td><td>1</td></tr>
<tr><td rowspan="3">3</td><td></td><td>$\Delta>0$</td><td>$\Delta<0$</td><td colspan="2" rowspan="2">Imaginary Cone</td><td colspan="2" rowspan="3">Impossible</td><td rowspan="3">Impossible</td></tr>
<tr><td>$T_2>0$; $A_{44}T_1>0$</td><td>Imaginary ellipsoid</td><td>Ellipsoid</td></tr>
<tr><td>$T_2>0$; $A_{44}T_1 \leqq 0$ or $T_2 \leqq 0$</td><td>Hyperboloid of one sheet</td><td>Hyperboloid of two sheets</td><td colspan="2">Real Cone</td></tr>
<tr><td rowspan="3">2</td><td colspan="2" rowspan="3">Hyperbolic paraboloid</td><td rowspan="3">Elliptic paraboloid</td><td>$T_2>0$; $T_1D_3>0$</td><td>Imaginary elliptic cylinder</td><td rowspan="2">$T_2>0$</td><td rowspan="2">Imaginary intersecting planes</td><td rowspan="3">Impossible</td></tr>
<tr><td>$T_2>0$; $T_1D_3<0$</td><td>Elliptic cylinder</td></tr>
<tr><td>$T_2<0$</td><td>Hyperbolic cylinder</td><td>$T_2<0$</td><td>Intersecting planes</td></tr>
<tr><td rowspan="2">1</td><td colspan="3" rowspan="2">Impossible</td><td colspan="2" rowspan="2">Parabolic cylinder</td><td>$D_2>0$</td><td>Imaginary parallel planes</td><td rowspan="2">Coincident planes</td></tr>
<tr><td>$D_2<0$</td><td>Parallel planes</td></tr>
</table>

In Theorem 16, Chapter VII (see Section 87, page 185) we proved that the quadric surfaces for which $r_4 = 4$ and $r_3 = 3$ have a single proper asymptotic cone; and that the surfaces for which $r_4 = 3$ and $r_3 = 2$ have a pair of asymptotic planes. In either case the asymptotic quadric of the surface $Q(x, y, z) = 0$ is given by the equation $Q(x, y, z) - Q(\alpha, \beta, \gamma) = 0$, where α, β, γ are the coördinates of a center of the surface. Since the equations of a quadric and of its asymptotic cone differ therefore only in the constant term, the invariants A_{44}, T_2, and T_1, which depend on the coefficients of the second degree terms only, are the same for the two surfaces. It is clear then from the above table that the asymptotic cone of the ellipsoid is imaginary, whereas that of the hyperboloids is real; also that the asymptotic planes are real for the hyperbolic cylinder and imaginary for the elliptic cylinder.

If these observations are combined with the results of Sections 84 and 85 we obtain the complete geometric characterization of the real quadric surfaces indicated in the table on page 232.

Examples.

1. To determine the character of the surface represented by the equation

$$5x^2 + 5y^2 + 8z^2 + 8yz + 8zx - 2xy + 12x - 12y + 6 = 0$$

we set up the matrices $\mathbf{a}_4$ and $\mathbf{a}_3$. We find

$$\mathbf{a}_4 = \begin{Vmatrix} 5 & -1 & 4 & 6 \\ -1 & 5 & 4 & -6 \\ 4 & 4 & 8 & 0 \\ 6 & -6 & 0 & 6 \end{Vmatrix} \quad \text{and} \quad \mathbf{a}_3 = \begin{Vmatrix} 5 & -1 & 4 \\ -1 & 5 & 4 \\ 4 & 4 & 8 \end{Vmatrix}.$$

The determinants of these matrices are both found to vanish because in each of them the third row is equal to the sum of the first and second rows. The third order principal minor of $\mathbf{a}_4$ which is formed from its last three rows and columns, and the two-rowed principal minor of $\mathbf{a}_3$ in its upper left-hand corner are both found to be different from zero. We conclude therefore that $r_4 = 3$ and $r_3 = 2$, and that the surface is a cylinder. Its axis, that is, its line of centers, is determined by any two of the system of three linear equations whose augmented matrix furnishes the first three rows of $\mathbf{a}_4$; we can take for it the equations $5x - y + 4z + 6 = 0$ and $x + y + 2z = 0$.

Moreover,

$$T_1 = 5 + 5 + 8 = 18, \quad T_2 = \begin{vmatrix} 5 & -1 \\ -1 & 5 \end{vmatrix} + \begin{vmatrix} 5 & 4 \\ 4 & 8 \end{vmatrix} + \begin{vmatrix} 5 & 4 \\ 4 & 8 \end{vmatrix} = 72;$$

and

$$D_3 = \begin{vmatrix} 5 & -1 & 4 \\ -1 & 5 & 4 \\ 4 & 4 & 8 \end{vmatrix} + \begin{vmatrix} 5 & -1 & 6 \\ -1 & 5 & -6 \\ 6 & -6 & 6 \end{vmatrix} + \begin{vmatrix} 5 & 4 & 6 \\ 4 & 8 & 0 \\ 6 & 0 & 6 \end{vmatrix} + \begin{vmatrix} 5 & 4 & -6 \\ 4 & 8 & 0 \\ -6 & 0 & 6 \end{vmatrix} = -432;$$

Hence, $T_2 > 0$ and $T_1 D_3 < 0$, so that the surface is an elliptic cylinder.

	Centers	Lines on surface	Asymptotic cone
Ellipsoid	Single proper center	No lines	Imaginary, proper
Hyperboloid of one sheet	Single proper center	Two lines through every point	Real, proper
Hyperboloid of two sheets	Single proper center	No lines	Real, proper
Hyperbolic paraboloid	No center	Two lines through every point	
Elliptic paraboloid	No center	No lines	
Cone	Single vertex		
Elliptic cylinder	Line of proper centers	Two coincident lines through every point	Imaginary, degenerate
Hyperbolic cylinder	Line of proper centers	Two coincident lines through every point	Real, degenerate
Parabolic cylinder	No center	Two coincident lines through every point	
Intersecting planes	Line of vertices		
Parallel planes	Plane of proper centers		
Coincident planes	Plane of vertices		

The discriminating equation is $k^3 - 18\,k^2 + 72\,k = 0$; the discriminating numbers are therefore 0, 6, 12. We put $k_1 = 0$, $k_2 = 6$, $k_3 = 12$. We know from the general discussion that rotation to principal directions will reduce the equation to the form $6\,y'^2 + 12\,z'^2 + 2\,a_{24}'y' + 2\,a_{34}'z' + 6 = 0$. To verify this fact, we proceed to determine the principal directions.

From $k_1 = 0$, we find $5\,\lambda_1 - \mu_1 + 4\,\nu_1 = 0$ and $-\lambda_1 + 5\,\mu_1 + 4\,\nu_1 = 0$, so that $\lambda_1 : \mu_1 : \nu_1 = 1 : 1 : -1$; from $k_2 = 6$, we find $-\lambda_2 - \mu_2 + 4\,\nu_2 = 0$ and $2\,\lambda_2 + 2\,\mu_2 + \nu_2 = 0$, so that $\lambda_2 : \mu_2 : \nu_2 = 1 : -1 : 0$; from $k_3 = 12$, we find $-7\,\lambda_3 - \mu_3 + 4\,\nu_3 = 0$ and $-\lambda_3 - 7\,\mu_3 + 4\,\nu_3 = 0$, so that $\lambda_3 : \mu_3 : \nu_3 = 1 : 1 : 2$. The equations for the rotation of axes to principal directions are therefore $x = \dfrac{x'}{\sqrt{3}} + \dfrac{y'}{\sqrt{2}} + \dfrac{z'}{\sqrt{6}}$, $y = \dfrac{x'}{\sqrt{3}} - \dfrac{y'}{\sqrt{2}} + \dfrac{z'}{\sqrt{6}}$, $z = -\dfrac{x'}{\sqrt{3}} + \dfrac{2\,z'}{\sqrt{6}}$.

If these expressions are substituted for x, y, and z in the original equation of the surface, it becomes

$$6\,y'^2 + 12\,z'^2 + 12\,y'\sqrt{2} + 6 = 0;$$

completing the square on the terms in y', we obtain as the final form of the equation $y''^2 + 2\,z''^2 = 1$, where $y'' = y' + \sqrt{2}$ and $z'' = z'$. The locus of this equation is indeed an elliptic cylinder; its axis is parallel to the X''-axis, i.e., to the line $y'' = 0$, $z'' = 0$. But the equations of this line may also be written in the form $y' + \sqrt{2} = 0$, $z' = 0$, or, with respect to the original frame of reference, in the form $x - y + 2 = 0$, $x + y + 2\,z = 0$, which is equivalent to the form of the equations of the axis of the cylinder given at the end of the first paragraph. The directrix of the cylinder is the ellipse $y''^2 + 2\,z''^2 = 1$, $x'' = 0$; the equations of this ellipse with respect to the original axes are $5\,x^2 + 5\,y^2 + 8\,z^2 + 8\,yz + 8\,zx - 2\,xy + 12\,x - 12\,y + 6 = 0, x+y-z=0.$

2. We proceed in a similar manner with the equation

$$2\,x^2 + 20\,y^2 + 18\,z^2 - 12\,yz + 12\,xy + 22\,x + 6\,y - 2\,z + 2 = 0.$$

We find $\mathbf{a}_4 = \begin{Vmatrix} 2 & 6 & 0 & 11 \\ 6 & 20 & -6 & 3 \\ 0 & -6 & 18 & -1 \\ 11 & 3 & -1 & 2 \end{Vmatrix}$ and $\mathbf{a}_3 = \begin{Vmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & -6 & 18 \end{Vmatrix}$;

$r_3 = 2$, $r_4 = 4$; $\Delta = -33, 124 < 0$. Therefore the surface represented by the given equation is an elliptic paraboloid.

Furthermore, $T_1 = 40$ and $T_2 = 364$ and the discriminating equation is $k^3 - 40\,k^2 + 364\,k = 0$; its roots are 0, 14, and 26. From the discussion of Case II (see Section 99, page 218), we know moreover that $a_{14}' = \pm\sqrt{-\dfrac{\Delta}{T_2}} = \sqrt{91}$. The equation of the surface is therefore reducible to the form $14\,y''^2 + 26\,z''^2 = \pm 2\sqrt{91}\,x''$.

This completes the determination of the type of surface represented by the equation and also of the numerical data necessary to fix its individuality. If we wish to determine its position with respect to the original frame of reference, we have to find the principal directions and also the new origin to which the axes have been translated.

From $k_1 = 0$, we find $\lambda_1 : \mu_1 : \nu_1 = 9 : -3 : -1$; from $k_2 = 14$, follows $\lambda_2 : \mu_2 : \nu_2 = 1 : 2 : 3$, and from $k_3 = 26$, we derive $\lambda_3 : \mu_3 : \nu_3 = 1 : 4 : -3$. If we base the rotation of axes to principal directions on the table

	X	Y	Z
X'	$\frac{9}{\sqrt{91}}$	$\frac{-3}{\sqrt{91}}$	$\frac{-1}{\sqrt{91}}$
Y'	$\frac{1}{\sqrt{14}}$	$\frac{2}{\sqrt{14}}$	$\frac{3}{\sqrt{14}}$
Z'	$\frac{1}{\sqrt{26}}$	$\frac{4}{\sqrt{26}}$	$\frac{-3}{\sqrt{26}}$

the equation is transformed into $14\,y'^2 + 26\,z'^2 + 2\sqrt{91}\,x' + 2\sqrt{14}\,y' + 2\sqrt{26}\,z' + 2 = 0$; translation of axes to the point $x' = 0$, $y' = \dfrac{-1}{\sqrt{14}}$, $z' = \dfrac{-1}{\sqrt{26}}$ leads to the final equation $14\,y''^2 + 26\,z''^2 = -2\sqrt{91}\,x''$. If the direction cosines of the X'-axis are changed in sign, that is, if its direction is reversed, the right-hand side of the final equation of the surface would be $2\sqrt{91}\,x''$. The surface extends indefinitely on the negative side of the plane $x'' = 0$, that is, in terms of the original system of coördinates, on that side of the plane $9\,x - 3\,y - z = 0$ which is determined by the direction cosines $\lambda = \dfrac{-9}{\sqrt{91}}$, $\mu = \dfrac{3}{\sqrt{91}}$, $\nu = \dfrac{1}{\sqrt{91}}$; it has the point $x' = 0$, $y' = \dfrac{-1}{\sqrt{14}}$, $z' = \dfrac{-1}{\sqrt{26}}$, that is, the point $x = -\dfrac{10}{91}$, $y = -\dfrac{27}{91}$, $z = -\dfrac{9}{91}$ in common with this plane.

103. Exercises.

Determine the type of quadric surface which is represented by each of the following equations and set up the reduced form of these equations; discuss their position in space in those cases in which the numerical work involved does not become too laborious:

1. $x^2 + 4\,y^2 + 9\,z^2 + 4\,xy + 6\,xz + 12\,yz - x + 2\,y + 5\,z = 0.$
2. $x^2 + z^2 + 2\,xy + 2\,xz - 2\,yz - 2\,x + 4\,y - 4 = 0.$
3. $x^2 + y^2 + z^2 + yz + zx + xy + 2\,x + 2\,y + 2\,z + \frac{3}{2} = 0.$
4. $4\,x^2 + y^2 + z^2 - 4\,xy + 6\,z + 8 = 0.$
5. $x^2 + y^2 + z^2 + yz + zx + xy + x + y + z + 1 = 0.$
6. $x^2 + y^2 + 2\,z^2 - 4\,xz + 2\,xy + 1 = 0.$
7. $z^2 - xy + x = 0.$
8. $5\,x^2 - y^2 + z^2 + 6\,zx + 4\,xy + 2\,x + 2\,y + 2\,z = 0.$
9. $x^2 + y^2 + z^2 - xy - yz - y = 0.$
10. $5\,x^2 + 13\,y^2 + 23\,z^2 + 36\,yz + 22\,zx + 16\,xy + 10\,x + 16\,y + 22\,z + 5 = 0.$
11. $x^2 + 5\,y^2 + 9\,z^2 + 4\,xy + 6\,yz - z - 3\,x = 0.$
12. $4\,y^2 + 4\,z^2 + 4\,yz - 2\,x - 14\,y - 22\,z + 33 = 0.$
13. $2\,y^2 + 4\,zx + 2\,x - 4\,y + 6\,z + 5 = 0.$
14. $x^2 + y^2 + z^2 - 6\,xy + 2\,zx - 6\,yz - 6\,x - 2\,y - 6\,z + 1 = 0.$
15. $3\,x^2 + 3\,y^2 + 3\,z^2 + 2\,yz + 2\,zx + 2\,xy + 1 = 0.$
16. $x^2 + xy + yz + zx - 3\,x - 2\,y - z - 3 = 0.$
17. $x^2 + 3\,yz - z^2 + 2\,x = 0.$
18. $(x - 2\,y + z)^2 + 4\,x - 8\,y + 4\,z + 3 = 0.$
19. $36\,x^2 + 4\,y^2 + z^2 - 4\,yz - 12\,zx + 24\,xy + 4\,x + 16\,y - 26\,z - 3 = 0.$
20. $x^2 + 4\,y^2 + z^2 - 4\,yz + 2\,zx - 4\,xy - 2\,x + 4\,y - 2\,z - 3 = 0.$
21. $2\,x^2 - 2\,yz + 2\,zx - 2\,xy - x - 2\,y + 3\,z - 2 = 0.$
22. Determine the condition which must be satisfied by the discriminating numbers of a quadric surface in order that it may be a surface of revolution; also the conditions which will insure that the surface is a sphere.
23. Set up the conditions which the invariants of the surface $Q(x, y, z) = 0$ must satisfy in order that the surface may be a sphere.

CHAPTER IX

QUADRIC SURFACES, SPECIAL PROPERTIES AND METHODS

In this chapter we shall discuss some properties and methods which are concerned with one or another of the classes of quadric surfaces, with which we have become acquainted.

104. The Reguli on the Hyperboloid of One Sheet. We have seen in Sections 84 and 102 that the hyperboloid of one sheet and the hyperbolic paraboloid are the only real quadric surfaces through every point of which there pass two real lines which lie entirely on the surface. The general method developed in Section 84, by means of which the existence of these lines was demonstrated, is not very convenient for actually determining the rulings through a particular point on a given surface; we shall therefore in this section take up a special method, for the hyperboloid of one sheet, for the solution of this problem. And, having shown in the preceding chapter that every quadric surface can be represented, with suitable choice of the frame of reference, by an equation characteristic of the type to which the surface belongs, we shall henceforth make use of these standard forms of the equations of the quadric surfaces.

The standard form of the equation of the hyperboloid of one sheet is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

It may be written in the form

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = \left(1 - \frac{z}{c}\right)\left(1 + \frac{z}{c}\right).$$

It should be easy to see from this that the line which is determined by the pair of equations

$$(1) \quad p_1\left(\frac{x}{a} - \frac{y}{b}\right) = p_2\left(1 - \frac{z}{c}\right) \quad \text{and} \quad p_2\left(\frac{x}{a} + \frac{y}{b}\right) = p_1\left(1 + \frac{z}{c}\right),$$

in which p_1 and p_2 are arbitrary real numbers which do not both vanish, lies entirely on the surface. Since the value of the ratio $p_1 : p_2$ may therefore be chosen arbitrarily, we have here a single infinitude of lines, or a **one-parameter family of lines** on the surface. This family of lines is of the type known in Projective Geometry as a **regulus.*** We shall use this term for convenience of reference without entering further into its definition; and we shall refer to the regulus whose lines are determined by equations (1) as the **p-regulus** of the surface.

It is moreover clear that we can obtain a second regulus of the surface in the form

$$(2)\quad q_1\left(\frac{x}{a}-\frac{y}{b}\right)=q_2\left(1+\frac{z}{c}\right),\quad q_2\left(\frac{x}{a}+\frac{y}{b}\right)=q_1\left(1-\frac{z}{c}\right);$$

we shall call this the **q-regulus,** it being again understood that q_1 and q_2 are arbitrary real numbers which do not both vanish.

A particular line l of the p-regulus is known as soon as the ratio $p_1 : p_2$ is given, and we shall designate this line by the symbol $l(p_1, p_2)$; similarly the symbol $m(q_1, q_2)$ will be used to designate a line of the q-regulus. To determine the particular line of each regulus which passes through a given point $A(\alpha, \beta, \gamma)$ on the surface, we substitute the coördinates of this point in one of the equations (1) to determine $p_1 : p_2$, and also in one of the equations (2) to determine $q_1 : q_2$.

Example.

To determine the lines through the point $A(4, -2, 3)$ on the surface $\frac{x^2}{16}+\frac{y^2}{4}-\frac{z^2}{9}=1$, we write the equations of the reguli; for the particular surface under consideration, these are:

$$p_1\left(\frac{x}{4}-\frac{z}{3}\right)=p_2\left(1-\frac{y}{2}\right),\quad p_2\left(\frac{x}{4}+\frac{z}{3}\right)=p_1\left(1+\frac{y}{2}\right);$$

and

$$q_1\left(\frac{x}{4}-\frac{z}{3}\right)=q_2\left(1+\frac{y}{2}\right),\quad q_2\left(\frac{x}{4}+\frac{z}{3}\right)=q_1\left(1-\frac{y}{2}\right).$$

Substituting the coördinates of A for x, y, z in these equations gives $p_1 \times 0 = p_2 \times 2$, $p_2 \times 2 = p_1 \times 0$; $q_1 \times 0 = q_2 \times 0$, $q_2 \times 2 = q_1 \times 2$. Therefore $p_2 = 0$ and p_1 is arbitrary and $q_1 = q_2$.

* For a definition and treatment of the regulus, see, for example, Veblen and Young, Projective Geometry, Vol. 1, pages 298–304.

Hence the required lines are given by the following pairs of equations:

$$\frac{x}{4} - \frac{z}{3} = 0, \quad 1 + \frac{y}{2} = 0; \quad \text{and} \quad \frac{x}{4} - \frac{z}{3} = 1 + \frac{y}{2}, \quad \frac{x}{4} + \frac{z}{3} = 1 - \frac{y}{2}.$$

105. Reguli on the Hyperboloid of One Sheet, *continued.* A number of questions concerning the reguli on the hyperboloid of one sheet, which suggest themselves naturally, will now be considered:

(1) Will the determination of $p_1 : p_2$ and of $q_1 : q_2$ always be possible in one and only one way?
(2) Do we get two distinct lines through every point on the surface?
(3) Do two lines of one regulus ever lie in the same plane?
(4) Will every line of the p-regulus be coplanar with every line of the q-regulus?
(5) What is the relative position of the plane determined by the two rulings which pass through a point A on the surface, and the tangent plane to the surface at P?

These questions, except the first, fall within the scope of Chapter IV and can be answered by the methods developed there. We shall take up these questions in some detail, because they furnish an opportunity to illustrate the application of these methods.

(1) *Will the determination of $p_1 : p_2$ and of $q_1 : q_2$ always be possible uniquely?* From an equation of the form $ap_1 = bp_2$ the ratio $p_1 : p_2$ is always uniquely determinable, unless $a = b = 0$. Hence if the point $A(\alpha, \beta, \gamma)$ lies on the hyperboloid of one sheet represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and if $\frac{\alpha}{a} - \frac{\beta}{b}, 1 - \frac{\gamma}{c}, \frac{\alpha}{a} + \frac{\beta}{b}, 1 + \frac{\gamma}{c}$ are not all zero, then at least one and at most two determinations of each of the ratios $p_1 : p_2$ and $q_1 : q_2$ are possible. But if these four expressions all vanished, it would follow that $\alpha = \beta = \gamma = 0$, which is not a point on the surface. And if the two equations (1) of Section 104 gave rise to two different values of $p_1 : p_2$, it would follow that $1 - \frac{\gamma}{c} : \frac{\alpha}{a} - \frac{\beta}{b} \neq \frac{\alpha}{a} + \frac{\beta}{b} : 1 + \frac{\gamma}{c}$, and hence that $\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} \neq 1 - \frac{\gamma^2}{c^2}$, so that the point $A(\alpha, \beta, \gamma)$

could not be on the surface. The same conclusion holds for the equations (2). The question is therefore to be answered affirmatively; consequently the method of the preceding section is always effective to determine the lines whose existence was proved in Section 84.

(2) *Will the two lines through $A(\alpha, \beta, \gamma)$ determined by the method of Section 104 always be distinct?*

If these lines are not distinct, then the four planes represented by the equations (1) and (2) of Section 104 for the particular values of $p_1 : p_2$ and $q_1 : q_2$ determined as in (1), must have a line in common. According to Theorem 23, Chapter IV (Section 54, page 101) this will happen if and only if the ranks of the coefficient matrix and the augmented matrix of their equations are both equal to 2.

The coefficient matrix of these equations is $\left\|\begin{matrix} \frac{p_1}{a} & -\frac{p_1}{b} & \frac{p_2}{c} \\ \frac{p_2}{a} & \frac{p_2}{b} & -\frac{p_1}{c} \\ \frac{q_1}{a} & -\frac{q_1}{b} & -\frac{q_2}{c} \\ \frac{q_2}{a} & \frac{q_2}{b} & \frac{q_1}{c} \end{matrix}\right\|$;

its rank is not affected by elementary transformations (compare Definition XIV and the Corollary of Theorem 14, Chapter I, Section 10, pages 18, 19), so that we may multiply its 1st, 2nd, and 3rd columns by a, b, and c respectively, and add the 1st column to the 2nd. This leads to the matrix $\left\|\begin{matrix} p_1 & 0 & p_2 \\ p_2 & 2\,p_2 & -p_1 \\ q_1 & 0 & -q_2 \\ q_2 & 2\,q_2 & q_1 \end{matrix}\right\|$.

It is found that the values of the third order determinants obtained from this matrix by omitting the 4th, 3rd, 2nd, and 1st rows respectively are equal to $-2\,p_2(p_1q_2 + p_2q_1)$, $2\,p_1(p_1q_2 + p_2q_1)$, $2\,q_2(p_1q_2 + p_2q_1)$ and $-2\,q_1(p_1q_2 + p_2q_1)$. Therefore, since, in accordance with (1), p_1, p_2, q_1, and q_2 never vanish simultaneously, the rank of the coefficient matrix can be 2 only if $p_1q_2 + p_2q_1 = 0$.

Let us now consider the matrix formed from the 1st, 3rd, and 4th columns of the augmented matrix; after applying to it similar

transformations as to the coefficient matrix we have to consider the matrix $\begin{Vmatrix} p_1 & 2p_2 & p_2 \\ p_2 & 0 & p_1 \\ q_1 & 0 & q_2 \\ q_2 & 2q_1 & q_1 \end{Vmatrix}$; the values of its third order determinants are found to be $-2p_2(p_2q_2 - p_1q_1)$, $2p_1(p_2q_2 - p_1q_1)$, $2q_2(p_2q_2 - p_1q_1)$ and $-2q_1(p_2q_2 - p_1q_1)$. Hence if the rank of the augmented matrix is also 2, we must have $p_2q_2 - p_1q_1 = 0$ as well as $p_1q_2 + p_2q_1 = 0$. If we look upon these equations as linear homogeneous equations in q_2 and q_1, we find, by using Theorem 2, Chapter II (Section 22, page 38), that either $q_1 = q_2 = 0$ or $p_1^2 + p_2^2 = 0$, that is $p_1 = p_2 = 0$. Neither of these cases can arise, in virtue of the discussion in (1). The answer to our second question is therefore also affirmative.

(3) *Do two lines of one regulus ever lie in one plane?*

Let us consider the lines $l(p_1, p_2)$ and $l'(p_1', p_2')$ of the p-regulus. They are given by the two pairs of equations

$$p_1\left(\frac{x}{a} - \frac{y}{b}\right) = p_2\left(1 - \frac{z}{c}\right), \qquad p_2\left(\frac{x}{a} + \frac{y}{b}\right) = p_1\left(1 + \frac{z}{c}\right);$$

and

$$p_1'\left(\frac{x}{a} - \frac{y}{b}\right) = p_2'\left(1 - \frac{z}{c}\right), \qquad p_2'\left(\frac{x}{a} + \frac{y}{b}\right) = p_1'\left(1 + \frac{z}{c}\right),$$

where $p_1 : p_2 \neq p_1' : p_2'$.

The value of the determinant of the augmented matrix of these equations is

$$\frac{1}{abc} \times \begin{vmatrix} p_1 & -p_1 & p_2 & -p_2 \\ p_2 & p_2 & -p_1 & -p_1 \\ p_1' & -p_1' & p_2' & -p_2' \\ p_2' & p_2' & -p_1' & -p_1' \end{vmatrix} = \frac{1}{abc} \times \begin{vmatrix} p_1 & 0 & p_2 & 0 \\ p_2 & 2p_2 & -p_1 & -2p_1 \\ p_1' & 0 & p_2' & 0 \\ p_2' & 2p_2' & -p_1' & -2p_1' \end{vmatrix}.$$

The evaluation of this determinant is accomplished most conveniently by means of the Laplace development of the 1st and 3rd columns (compare Theorem 15, Chapter I, Section 12, page 23); we find then that its value is $\begin{vmatrix} p_1 & p_2 \\ p_1' & p_2' \end{vmatrix} \times \begin{vmatrix} 2p_2 & -2p_1 \\ 2p_2' & -2p_1' \end{vmatrix} = 4(p_1p_2' - p_2p_1')^2$, which is different from zero, in virtue of the hypothesis $p_1 : p_2 \neq p_1' : p_2'$. It follows from this, on account of Theorem 22, Chapter IV, (Section 54, page 101), that the two lines

of the p-regulus can not have a point in common. Can they be parallel?

There should be no difficulty in showing that the direction cosines of the lines $l(p_1, p_2)$ and $l'(p_1', p_2')$ are given by the proportions

$$\lambda : \mu : \nu = a(p_1^2 - p_2^2) : b(p_1^2 + p_2^2) : 2\,cp_1p_2,$$

and

$$\lambda' : \mu' : \nu' = a(p_1'^2 - p_2'^2) : b(p_1'^2 + p_2'^2) : 2\,cp_1'p_2'.$$

Since p_1, p_2, p_1' and p_2' are all real, both $p_1^2 + p_2^2$ and $p_1'^2 + p_2'^2$ must be positive. Therefore, if the lines are to be parallel, there must exist a positive factor of proportionality ρ^2, such that

$$p_1^2 - p_2^2 = \rho^2(p_1'^2 - p_2'^2),\ p_1^2 + p_2^2 = \rho^2(p_1'^2 + p_2'^2) \text{ and } p_1p_2 = \rho^2 p_1'p_2'.$$

From the first two of these equations we conclude, by addition and subtraction, that $p_1^2 = \rho^2 p_1'^2$ and $p_2^2 = \rho^2 p_2'^2$, and hence that $p_1 = \pm\,\rho p_1'$ and $p_2 = \pm\rho p_2'$. If opposite signs were used in these two equations, it would follow that $p_1p_2 = -\rho^2 p_1'p_2'$, which is in conflict with the third of the above equations. Therefore the lines can have the same direction cosines only if $p_1 : p_2 = p_1' : p_2'$, that is, if they coincide.

Our discussion has therefore brought us to the conclusion that no two lines of the p-regulus can lie in the same plane.

The reader is urged to carry through the same argument for the q-regulus. When he has done this, he will have completed the proof that the answer to the third question is in the negative.

(4) *Will every line of the p-regulus be coplanar with every line of the q-regulus?*

The discussion of this question will be based on Theorem 25, Chapter IV (Section 54, page 104). If $l(p_1, p_2)$ is an arbitrary line of the p-regulus and $m(q_1, q_2)$ an arbitrary line of the q-regulus, these lines will be coplanar unless the determinant

$$\begin{vmatrix} \lambda & \lambda_1 & \alpha - \alpha_1 \\ \mu & \mu_1 & \beta - \beta_1 \\ \nu & \nu_1 & \gamma - \gamma_1 \end{vmatrix}$$

is different from zero; here λ, μ, ν and λ_1, μ_1, ν_1 are the direction cosines of the lines l and m respectively, and (α, β, γ) and $(\alpha_1, \beta_1, \gamma_1)$ are arbitrary points on these lines. We have already seen that $\lambda : \mu : \nu = a(p_1^2 - p_2^2) : b(p_1^2 + p_2^2) : 2\,cp_1p_2$ (see (3)

above); in similar manner we find that $\lambda_1 : \mu_1 : \nu_1 = a(q_1^2 - q_2^2) : b(q_1^2 + q_2^2) : -2\,cq_1q_2$. It remains therefore to investigate whether or not the determinant

$$\begin{vmatrix} a(p_1^2 - p_2^2) & a(q_1^2 - q_2^2) & \alpha - \alpha_1 \\ b(p_1^2 + p_2^2) & b(q_1^2 + q_2^2) & \beta - \beta_1 \\ 2\,cp_1p_2 & -2\,cq_1q_2 & \gamma - \gamma_1 \end{vmatrix}$$

has a value which is different from zero, when for α, β, γ and $\alpha_1, \beta_1, \gamma_1$ are substituted sets of numbers which satisfy the pairs of equations $p_1\left(\frac{x}{a} - \frac{y}{b}\right) = p_2\left(1 - \frac{z}{c}\right)$, $p_2\left(\frac{x}{a} + \frac{y}{b}\right) = p_1\left(1 + \frac{z}{c}\right)$ and $q_1\left(\frac{x}{a} - \frac{y}{b}\right) = q_2\left(1 + \frac{z}{c}\right)$, $q_2\left(\frac{x}{a} + \frac{y}{b}\right) = q_1\left(1 - \frac{z}{c}\right)$ respectively. It turns out that this determinant vanishes for every such choice of α, β, γ and α_1, β_1, γ_1; the details of the proof of this statement are given in Appendix, VII (page 299). Hence we conclude that every line of the p-regulus is coplanar with every line of the q-regulus.

(5) *What is the relative position of the plane determined by the two rulings which pass through a point on the surface and the tangent plane to the surface at this point?*

The equation of the tangent plane to the surface at a point $A(\alpha, \beta, \gamma)$ on the surface is

$$\frac{\alpha x}{a^2} - \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1$$

(compare Theorem 4, Chapter VII, Section 81, page 162). This plane and the line of the p-regulus through A have at least the point A in common; the line will therefore lie in the plane or meet it in a single point, according as

$$a(p_1^2 - p_2^2) \times \frac{\alpha}{a^2} - b(p_1^2 + p_2^2) \times \frac{\beta}{b^2} + 2\,cp_1p_2 \times \frac{\gamma}{c^2}$$

is equal to or different from zero (compare Theorem 21, Chapter IV, Section 52, page 99). This expression is equal to $p_1^2\left(\frac{\alpha}{a} - \frac{\beta}{b}\right) - p_2^2\left(\frac{\alpha}{a} + \frac{\beta}{b}\right) + 2\,p_1p_2\frac{\gamma}{c}$; and this reduces, by virtue of the equations of the lines of the p-regulus, to $p_1p_2\left(1 - \frac{\gamma}{c}\right) - p_2p_1\left(1 + \frac{\gamma}{c}\right)$

$+ 2 p_1 p_2 \frac{\gamma}{c} = 0$. Consequently, the line of the p-regulus lies in the tangent plane; the reader should have no difficulty in proving that the line of the q-regulus which passes through a given point on the surface also lies in the tangent plane to the surface at that point. These conclusions could also have been reached by a geometric discussion, namely, by observing that any line which connects A with a point on the plane determined by the lines $l(p_1, p_2)$ and $m(q_1, q_2)$ meets the surface in two points which coincide at A, and is therefore tangent to the surface at A (compare Definitions V and VI, Chapter VI, Section 77, pages 154, 155).

We summarize the results obtained in this section in a theorem.

THEOREM 1. **A hyperboloid of one sheet contains two one-parameter families of lines. Through every point of the surface passes one and only one line of each family. No two lines of the same family are coplanar; every line of one family is coplanar with every line of the other family. The plane determined by the two lines which pass through an arbitrary point on the surface is coincident with the tangent plane to the surface at this point.**

106. The Reguli on the Hyperbolic Paraboloid. The standard form to which the equation of an hyperbolic paraboloid can be reduced is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2\,nz.$$

If we write this equation in the form

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 2\,n \cdot z,$$

it becomes clear that the surface contains the two one-parameter families of lines represented by the following pairs of equations

$p_1\left(\frac{x}{a} - \frac{y}{b}\right) = p_2 z,\quad p_2\left(\frac{x}{a} + \frac{y}{b}\right) = 2\,np_1,\quad p_1$ and p_2 real numbers, not both zero;

$q_1\left(\frac{x}{a} - \frac{y}{b}\right) = 2\,nq_2,\quad q_2\left(\frac{x}{a} + \frac{y}{b}\right) = q_1 z,\quad q_1$ and q_2 real numbers, not both zero.

With reference to these reguli on the hyperbolic paraboloid, the same questions arise as were discussed for the hyperboloid of one

sheet. The discussion of these questions is left to the reader (see Section 108).

107. The Straight Lines on the Singular, Non-degenerate Quadrics. It was proved in Theorem 12, Chapter VII (Section 84, page 175), that through every point on a non-degenerate singular quadric which is not a vertex of the surface, there pass two coincident lines which lie entirely on the surface. This class of surfaces includes the proper cone and the cylinders.

If the equation of the proper cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

is written in the form

$$\left(\frac{x}{a}-\frac{z}{c}\right)\left(\frac{x}{a}+\frac{z}{c}\right)=-\frac{y^2}{b^2}$$

we recognize that the one-parameter families of lines, represented by the pairs of linear equations

$$p_1\left(\frac{x}{a}-\frac{z}{c}\right)=p_2\cdot\frac{y}{b},\qquad p_2\left(\frac{x}{a}+\frac{z}{c}\right)=-p_1\cdot\frac{y}{b};$$

and $$q_1\left(\frac{x}{a}-\frac{z}{c}\right)=-q_2\cdot\frac{y}{b}\qquad q_2\left(\frac{x}{a}+\frac{z}{c}\right)=q_1\cdot\frac{y}{b},$$

in which p_1, p_2, q_1, and q_2 are real numbers and neither p_1 and p_2, nor q_1 nor q_2 vanishes simultaneously, lie entirely on the surface. It should be clear however that the lines $l(p_1, p_2)$ and $m(q_1, q_2)$ are identical when $q_1 = p_1$ and $q_2 = -p_2$. Consequently, these two families of lines are identical; the proper cone contains therefore two coincident reguli.

A similar argument shows that the elliptic cylinder, the hyperbolic cylinder, and the parabolic cylinder, also each contain two coincident reguli.

108. Exercises.

1. Determine the equations of the straight lines on the surface $\frac{x^2}{4}-\frac{y^2}{9}=2z$, which pass (*a*) through the point $A(10, 9, 8)$; (*b*) through the point $B(-10, 9, 8)$; (*c*) through the point $C(10, -9, 8)$; (*d*) through the point $D(-10, -9, 8)$.

2. Determine the equations of the rulings of the surface $\frac{x^2}{4}+\frac{y^2}{9}-z^2=1$

which pass (*a*) through the point $A(4, -3, 2)$; (*b*) through the point $B(-4, 3, 2)$; (*c*) through the point $C(4, -3, -2)$; (*d*) through the point $D(-4, -3, -2)$.

3. Determine the equations of the straight lines on the cone $x^2 - y^2 + z^2 = 0$, which pass (*a*) through the point $A(3, -5, 4)$; (*b*) through the point $B(-3, 5, 4)$; (*c*) through the point $C(3, 5, 4)$; (*d*) through the point $D(-3, 5, -4)$.

4. Show that the elliptic cylinder, the hyperbolic cylinder and the parabolic cylinder each contain two coincident one-parameter families of lines.

5. Prove that any two lines which belong to one regulus of the hyperbolic paraboloid are skew.

6. Prove that every line of one regulus of the hyperbolic paraboloid is coplanar with every line of the other regulus of that surface.

7. Prove that the plane tangent to the hyperbolic paraboloid at an arbitrary point contains the two rulings of the surface which pass through that point.

8. Determine the cosine of the angles made by the two lines which pass through an arbitrary point of the hyperboloid of one sheet and determine the condition under which these two lines are perpendicular.

9. Discuss the corresponding question for the hyperbolic paraboloid.

10. Prove that to every point $A(\alpha, \beta, \gamma)$ on the hyperboloid of one sheet there corresponds a point A' on the surface such that the line of the p-regulus through A is parallel to the line of the q-regulus through A', and vice versa.

11. Determine the locus of all points on the hyperboloid of one sheet for which the angle between the rulings of the surface which pass through them is constant.

12. Show that it is possible to set up a correspondence between each of the reguli of the hyperboloid of one sheet on the one hand, and the regulus of its asymptotic cone on the other, such that corresponding lines are parallel.

109. Circles on Quadric Surfaces, the General Method. In Section 66 we discussed the problem of determining the curve of intersection of a plane and a surface. In accordance with Corollary 2 of Theorem 8, Chapter V (Section 66, page 128), a plane section of a quadric surface is a curve of degree not higher than the second. It is of interest to inquire whether plane sections of quadric surfaces can be circles. In answer to this question we shall prove in the first place the following theorem.

THEOREM 2. **The sections of a quadric surface made by two parallel planes are either both circles or else neither is a circle.**

Proof. The method developed in Section 66 for determining the character of a plane section of a surface consisted in rotating the axes in such a way as to make one of the new coördinate planes parallel to the plane of the section. The rotation required by this

method is the same for the sections of a surface by each of two parallel planes $\lambda x + \mu y + \nu z - p_1 = 0$ and $\lambda x + \mu y + \nu z - p_2 = 0$. Let us suppose that the rotation of axes is made in such manner as to make λ, μ, ν the direction cosines of the new Z-axis; then the equations of the two given planes will be, with reference to the new axes, $z' = p_1$ and $z' = p_2$. Consequently the plane equations of the two sections will be obtained when, in the new equation of the quadric surface, z' is replaced by p_1 and p_2. If the transformed equation of the quadric surface is $Q(x, y, z) = 0$, then the plane equations of the sections will be

$$\begin{aligned} &a_{11}x^2 + 2\,a_{12}xy + a_{22}y^2 + 2\,(a_{13}p_1 + a_{14})x + 2\,(a_{23}p_1 + a_{24})y \\ &\qquad + a_{33}p_1{}^2 + 2\,a_{34}p_1 + a_{44} = 0, \\ &a_{11}x^2 + 2\,a_{12}xy + a_{22}y^2 + 2\,(a_{13}p_2 + a_{14})x + 2\,(a_{23}p_2 + a_{24})y \\ &\qquad + a_{33}p_2{}^2 + 2\,a_{34}p_2 + a_{44} = 0. \end{aligned}$$

Since the condition that the plane locus of either of these equations shall represent a circle is that $a_{11} = a_{22}$ and $a_{12} = 0$, the theorem has been established.

Remark. To determine the circular sections of a quadric surface, it suffices to consider the planes through some fixed point.

We could now proceed for the further discussion of our problem, to determine the plane equation of the section of the quadric by an arbitrary plane through the origin and then impose the conditions which insure that the locus of this equation is a circle. But we have had opportunity to observe before that the most direct method is not always the most convenient and that the end we are seeking to accomplish is frequently reached in a more elegant and more instructive way by a more sophisticated procedure. This will be our program in the present case.

We begin by recalling from Elementary Plane Geometry that, if lines $l_1, l_2, \ldots$ are drawn through a point P in the plane of a circle, meeting the circle in pairs of points $A_1, B_1, A_2, B_2, \ldots$, then the products $PA_1 \cdot PB_1, PA_2 \cdot PB_2, \ldots$ are all equal to each other. It is true conversely, that if $A_1B_1, A_2B_2, \ldots$ are chords of a conic section which pass through a fixed point P and the products $PA_1 \cdot PB_1, PA_2 \cdot PB_2, \ldots$ are all equal for any fixed position of P in the plane of the conic section, then this conic is a circle.*

* A proof of this converse theorem will be found in Appendix, VIII, page 300.

Suppose now that the plane through the origin, whose equation is

$$(1)\quad ax + by + cz = 0$$

cuts the quadric surface Q in a circle and that the lines $l_1, l_2, l_3 \ldots$ through the origin which lie in this plane cut the surface in the pairs of points $A_1, B_1;\ A_2, B_2; \ldots$. If the direction cosines of an arbitrary one of these lines is designated by λ, μ, ν, these direction cosines satisfy the condition $a\lambda + b\mu + c\nu = 0$ (compare Theorem 21, Chapter IV, Section 52, page 99). And the distances from O to the points in which the line meets the quadric are given, in magnitude and direction, by the roots of the equation

$$q(\lambda, \mu, \nu)s^2 + 2\,(a_{14}\lambda + a_{24}\mu + a_{34}\nu)s + a_{44} = 0,$$

(compare Theorem 1, Chapter VII, Section 80, page 160 and remember that in this case $\alpha = \beta = \gamma = 0$); and the product of these roots is equal to $\dfrac{a_{44}}{q(\lambda, \mu, \nu)}\cdot$ It follows that if the section is a circle, then $\dfrac{a_{44}}{q(\lambda, \mu, \nu)}$ must have the same value for all those admissible values of λ, μ, ν for which $a\lambda + b\mu + c\nu = 0$. This means that there must exist a number k, which is independent of λ, μ, and ν, such that $q(\lambda, \mu, \nu) = k$, whenever $a\lambda + b\mu + c\nu = 0$, or again that the quadratic equation $q(\lambda, \mu, \nu) - k = 0$ is satisfied whenever the linear equation $a\lambda + b\mu + c\nu = 0$ is satisfied. From this we conclude, *first* that the quadratic function $q(\lambda, \mu, \nu) - k$ must be factorable in two linear functions of λ, μ, ν with real or complex coefficients (compare the argument made in the proof of Corollary 2 of Theorem 12, Chapter VII, Section 84, page 175); and *secondly*, that $a\lambda + b\mu + c\nu$ must be one of the factors. Furthermore, the function $q_k(\lambda, \mu, \nu) \equiv q(\lambda, \mu, \nu) - k = q(\lambda, \mu, \nu) - k(\lambda^2 + \mu^2 + \nu^2)$ is factorable, according to Corollary 3 of Theorem 8, Chapter VIII (Section 96, page 209) if and only if the rank of its discriminant matrix is less than 3, that is, if and only if

$$\begin{vmatrix} a_{11} - k & a_{12} & a_{13} \\ a_{12} & a_{22} - k & a_{23} \\ a_{13} & a_{23} & a_{33} - k \end{vmatrix} = 0.$$

But this equation is the discriminating equation $A(k) = 0$ of the surface Q (compare Sections 88 and 89). If, as before, we designate its roots by k_1, k_2, and k_3, we conclude that if the plane $ax + by + cz = 0$ cuts the quadric Q in a circle, then $ax + by + cz$ must be a factor of one of the quadratic functions $q(x, y, z) - k_i(x^2 + y^2 + z^2)$, $i = 1, 2$ or 3.

Conversely, if $ax + by + cz$ is a factor of $q(x, y, z) - k_i(x^2 + y^2 + z^2)$, for $i = 1, 2$, or 3, then $a\lambda + b\mu + c\nu$ is a factor of $q(\lambda, \mu, \nu) - k_i(\lambda^2 + \mu^2 + \nu^2)$, that is, of $q(\lambda, \mu, \nu) - k_i$; consequently $q(\lambda, \mu, \nu) = k_i$ for all admissible values of λ, μ, ν for which $a\lambda + b\mu + c\nu = 0$. If we take now an arbitrary point $P(\alpha, \beta, \gamma)$ in the plane $ax + by + cz = 0$, the lines in the plane through P will cut the quadric Q in pairs of points A, B whose distances from P are the roots of the equation $L_0s^2 + 2L_1s + L_2 = 0$, where $L_0 = q(\lambda, \mu, \nu)$ and $L_2 = Q(\alpha, \beta, \gamma)$ (see Theorem 1, Chapter VII, Section 80, page 160); hence the product $PA \cdot PB = \dfrac{Q(\alpha, \beta, \gamma)}{q(\lambda, \mu, \nu)} = \dfrac{Q(\alpha, \beta, \gamma)}{k_i}$ for all admissible values of λ, μ, ν for which $a\lambda + b\mu + c\nu = 0$. But this expresses the fact that the product of these distances is constant for all lines through P, no matter what point P is chosen, and therefore (see footnote on page 245) that the plane cuts the quadric in a circle. We have therefore established the following theorem.

THEOREM 3. **The necessary and sufficient condition that the plane through the origin represented by the equation $ax + by + cz = 0$ shall be a plane of circular section of the quadric surface $Q(x, y, z) = 0$ is that $ax + by + cz$ must be a factor of the homogeneous quadratic function $q(x, y, z) - k_i(x^2 + y^2 + z^2)$ for $i = 1, 2$ or 3, where k_1, k_2, and k_3 are the discriminating numbers of the surface.**

The discriminating numbers are all real (see Theorem 20, Chapter VII, Section 89, page 192), but they need not all be distinct. Moreover, even though k_1, k_2, and k_3 are real, it is not certain whether the linear factors of $q(x, y, z) - k_i(x^2 + y^2 + z^2)$ are real for $i = 1, 2, 3$, that is, whether the planes of circular section are real. In view of Theorems 2 and 3 we can state therefore that through every point in space there pass six planes which cut an arbitrary quadric surface in circles; of these planes some may be coincident and some may be imaginary. We proceed now to a further study of the different possibilities.

We suppose first that the equation $A(k) = 0$ has no multiple roots; then the rank of the matrix

$$\mathbf{a}_3(k_i) = \left\| \begin{matrix} a_{11} - k_i & a_{12} & a_{13} \\ a_{12} & a_{22} - k_i & a_{23} \\ a_{13} & a_{23} & a_{33} - k_i \end{matrix} \right\|$$

is equal to 2, for $i = 1, 2, 3$ (see Theorem 19, Chapter VII, Section 89, page 190) and hence the invariant $T_2(k_i)$ of the quadric surface represented by the equation $q(x, y, z) - k_i(x^2 + y^2 + z^2) = 0$ is different from zero. It follows therefore from the table in Section 102 (page 230) that this surface consists of two distinct real planes or two imaginary planes according as $T_2(k_i) < 0$ or $T_2(k_i) > 0$. But

$$\begin{aligned} T_2(k_i) &= \begin{vmatrix} a_{22}-k_i & a_{23} \\ a_{23} & a_{33}-k_i \end{vmatrix} + \begin{vmatrix} a_{11}-k_i & a_{13} \\ a_{13} & a_{33}-k_i \end{vmatrix} + \begin{vmatrix} a_{11}-k_i & a_{12} \\ a_{12} & a_{22}-k_i \end{vmatrix} \\ &= (a_{22} - k_i)(a_{33} - k_i) - a_{23}^2 + (a_{11} - k_i)(a_{33} - k_i) - a_{13}^2 \\ &\qquad + (a_{11} - k_i)(a_{22} - k_i) - a_{12}^2 \\ &= T_2 - 2\,T_1k_i + 3\,k_i^2 = -A'(k_i).^* \end{aligned}$$

Now $A'(k_i)$ represents the slope of the curve $y = A(k)$ at the points where it crosses the K-axis; since we are supposing that the equation $A(k) = 0$ has no multiple roots, and since the coefficient of k^3 in $A(k)$ is -1, the curve $y = A(k)$ has the general character indicated in Fig. 33. From this it should be evident that of the three numbers $A'(k_i)$, $i = 1, 2, 3$, two are negative, namely, those which correspond to the least and to the greatest of the numbers k_1, k_2, k_3, whereas one is positive, namely, the one which corresponds to the middle discriminating number; conse-

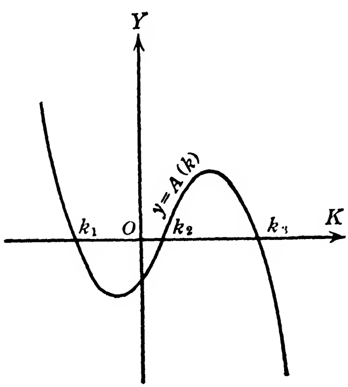

Fig. 33

* Notice that in $A(k)$ the coefficient of k^3 is -1. This interesting formula can be obtained directly from the proof of Theorem 19, Chapter VII (see Section 89, page 190); the alternate proof given in the text does not make use of the formula for the derivative of a determinant, but is not well suited for extension to derivatives of higher order.

quently, of the numbers $T_2(k_i)$, two are positive and one is negative, and if the notation be so chosen that $k_1 < k_2 < k_3$, $T_2(k_1) > 0$, $T_2(k_2) < 0$ and $T_2(k_3) < 0$. Therefore the equation $q(x, y, z) - k_2(x^2 + y^2 + z^2) = 0$ represents a pair of real planes of circular section through the origin, but the equations $q(x, y, z) - k_1(x^2 + y^2 + z^2) = 0$ and $q(x, y, z) - k_3(x^2 + y^2 + z^2) = 0$ represent pairs of imaginary planes.

If the equation $A(k) = 0$ has a pair of double roots, let us say $k_1 = k_2$, then the rank of the matrix $\mathbf{a}_3(k_1)$ is 1 (see Theorem 19, Chapter VII) and therefore the equation $q(x, y, z) - k_1(x^2+y^2+z^2) = 0$ represents a pair of coincident planes (see Corollary 3 of

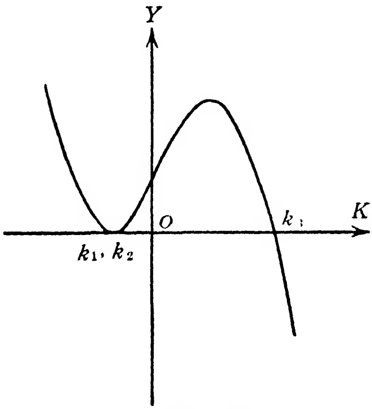

FIG. 34a

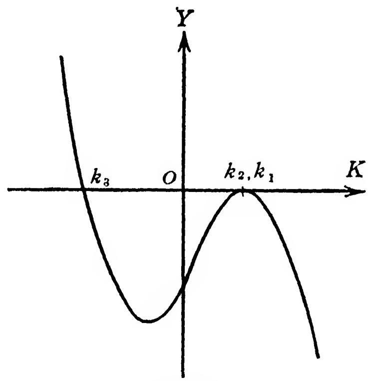

FIG. 34b

Theorem 8, Section 96, page 209). From the discussion in the preceding paragraph we conclude that in this case the graph of the function $A(k)$ has the character indicated in Figs. 34a and 34b, from which it should be clear that $A'(k_3) < 0$, therefore that $T_2'(k_3) > 0$ and hence that the planes represented by the equation $q(x, y, z) - k_3(x^2 + y^2 + z^2) = 0$ are imaginary. In this case there are therefore four coincident planes through the origin which cut the surface in a circle; each of these planes is represented by the equation $[q(x, y, z) - k_1(x^2 + y^2 + z^2)]^{\frac{1}{2}} = 0$.

Finally, if k_1 is a triple root of the equation $A(k) = 0$, the rank of the matrix $\mathbf{a}_3(k_1)$ is 0 (see Theorem 19, Chapter VII) and therefore the function $q(x, y, z) - k_1(x^2 + y^2 + z^2)$ vanishes identically, so that every function of the form $ax + by + cz$ is a factor of it. In this case every plane through the origin is a plane of circular

section. In view of the remark on page 215 this constitutes a proof of the well-known fact that the section of a sphere by an arbitrary plane is a circle.

We should recall moreover that if the discriminating equation has a pair of equal roots which are not zero, the quadric is a surface of revolution, while if it has a pair of zero roots, the quadric is a parabolic cylinder or a pair of parallel or coincident planes. In view of these facts we can state the following conclusion:

THEOREM 4. **The quadrics which are surfaces of revolution but not spheres, the parabolic cylinder, and the pair of parallel or coincident planes possess through every point of space $A(\alpha, \beta, \gamma)$ four coincident planes of circular section; in these cases there exists a double root, k_1, of the discriminating equation, the function $q(x, y, z) - k_1(x^2 + y^2 + z^2)$ is a perfect square and the planes of circular section through A are given by the equation $[g(x - \alpha, y - \beta, z - \gamma)]^4 = 0$, where $[g(x, y, z)]^2 = q(x, y, z) - k_1(x^2 + y^2 + z^2)$. All other quadric surfaces possess through every point of space $A(\alpha, \beta, \gamma)$ two distinct planes of circular section; no two of the discriminating numbers k_1, k_2, k_3 are equal to each other and, if $k_1 < k_2 < k_3$, the function $q(x, y, z) - k_2(x^2 + y^2 + z^2)$ is factorable into two linear factors with real coefficients; if we write $q(x, y, z) - k_2(x^2 + y^2 + z^2) = g_1(x, y, z) \cdot g_2(x, y, z)$, the planes of circular section through A are given by the equations $g_1(x - \alpha, y - \beta, z - \gamma) = 0$ and $g_2(x - \alpha, y - \beta, z - \gamma) = 0$.**

Remark 1. The circles of these circular sections may be ordinary circles with a finite center and finite radius, or they may be "degenerate circles" (compare Appendix, VIII, page 301). It should be clear that the circular sections of degenerate quadrics are always degenerate circles. And it should be clear that this will also be the case when the middle root or the double root of the discriminating equation is equal to zero (compare also Sections 110 and 111).

Remark 2. If the notation for the discriminating numbers is so chosen that in all cases $k_1 \leqq k_2 \leqq k_3$, the planes of circular section through the origin are always given by the equation $q(x, y, z) - k_2(x^2 + y^2 + z^2) = 0$.

Example.

The locus of the equation

$$3x^2 - y^2 - z^2 + 6yz - 6x + 4y - 2z - 2 = 0$$

is an hyperboloid of one sheet; for $\Delta = \begin{vmatrix} 3 & 0 & 0 & -3 \\ 0 & -1 & 3 & 2 \\ 0 & 3 & -1 & -1 \\ -3 & 2 & -1 & -2 \end{vmatrix} = 99,$

$A_{44} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{vmatrix} = -24$, $T_2 = -14$, so that $\Delta > 0$ and $T_2 < 0$. Moreover we find that $T_1 = 1$. The discriminating equation is $k^3 - k^2 - 14k + 24 = 0$; its roots are 2, 3, and -4. Therefore in the notation of Remark 2 above, $k_2 = 2$, and the planes through the origin which cut the surface in circles are given by the equation $3x^2 - y^2 - z^2 + 6yz - 2(x^2 + y^2 + z^2) = 0$, that is, by $x^2 - 3(y - z)^2 = 0$ or by $x - \sqrt{3}\,y + \sqrt{3}\,z = 0$ and $x + \sqrt{3}\,y - \sqrt{3}\,z = 0$. The planes of circular section through an arbitrary point $A(\alpha, \beta, \gamma)$ are given by the equations $(x - \alpha) - \sqrt{3}(y - \beta) + \sqrt{3}(z - \gamma) = 0$ and $(x - \alpha) + \sqrt{3}(y - \beta) - \sqrt{3}(z - \gamma) = 0$.

110. Circles on Quadric Surfaces, *continued.* To determine the planes of circular section for a particular quadric surface, whose equation is given in numerical form, we can proceed by the general method developed in the preceding section. The work becomes very simple if the equation of the surface has first been reduced to the standard forms of Sections 100–102.

Examples.

1. If in the equation of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, $a < b < c$, the middle discriminating number of the surface is $\dfrac{1}{b^2}$. Therefore the planes of circular section through the origin are given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{1}{b^2}(x^2 + y^2 + z^2) = 0,$$

that is, by
$$x^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) - z^2\left(\frac{1}{b^2} - \frac{1}{c^2}\right) = 0.$$

In virtue of the relative magnitude of a, b, and c, which has been presupposed, the coefficients of x^2 and z^2 in this equation are both positive. Hence the planes of circular section through the origin are represented by the equations

$$cx\sqrt{b^2 - a^2} - az\sqrt{c^2 - b^2} = 0 \quad \text{and} \quad cx\sqrt{b^2 - a^2} + az\sqrt{c^2 - b^2} = 0;$$

and the planes of circular section through the arbitrary point $A(\alpha, \beta, \gamma)$ are given by the equations

$$c\sqrt{b^2 - a^2}(x - \alpha) = a\sqrt{c^2 - b^2}(z - \gamma) \quad \text{and} \quad c\sqrt{b^2 - a^2}(x - \alpha) = -a\sqrt{c^2 - b^2}(z - \gamma).$$

2. For the parabolic cylinder $y^2 = 4pz$, the discriminating numbers are 0, 0, 1; therefore $k_2 = 0$ and the planes of circular section are given by the equation $y^4 = 0$. This equation represents the ZX-plane counted fourfold. The planes of circular section through the point $A(\alpha, \beta, \gamma)$ are represented by the equation $(y - \beta)^4 = 0$; therefore they are the planes $y = \beta$ counted fourfold. The intersection of a plane of this family consists of the generating

line $y = \beta$, $z = \frac{\beta^2}{4p}$ and of the infinitely distant line of the plane $y = \beta$; the circular section is therefore a degenerate circle (compare Appendix, page 301).

111. Exercises.

1. Determine the planes of circular section through the point $A(-2, 3, 1)$ for each of the following surfaces:

$$(a)\ \frac{x^2}{4} - \frac{y^2}{9} - z^2 = 1, \qquad (b)\ -\frac{x^2}{4} + \frac{y^2}{2} + \frac{z^2}{4} = 1,$$

$$(c)\ \frac{x^2}{4} + \frac{y^2}{9} = 2z, \qquad (d)\ \frac{x^2}{4} - \frac{y^2}{9} = 12z.$$

2. Prove that the circular sections of a hyperbolic paraboloid are always degenerate.

3. Prove that the two families of planes of circular section of a central quadric are not affected when the surface is translated.

4. Determine the planes of circular section through the point $A(3, 4, -1)$ for each of the following surfaces:

$$(a)\ x^2 + 3y^2 - z^2 = 0, \qquad (b)\ 4x^2 + 9y^2 = 1, \qquad (c)\ 4x^2 - 9y^2 = 1.$$

5. Prove that the circular sections of the hyperbolic cylinder are always degenerate.

6. Determine the angle between the two planes through the origin which cut the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in circles; and set up the condition under which these planes will be perpendicular.

7. Solve the similar problem for the hyperboloid of one sheet and also for the hyperboloid of two sheets.

8. Determine the planes of circular section through the point $A(2, -1, 1)$ for each of the following surfaces:

$$(a)\ 4x^2 + 6y^2 + 4z^2 = 1, \qquad (b)\ x^2 - 2y^2 + z^2 = 1,$$

$$(c)\ 2x^2 - y^2 - z^2 = 1, \qquad (d)\ 4x^2 + 4y^2 = 5z,$$

$$(e)\ 6x^2 - y^2 + 6z^2 = 0.$$

9. Prove that for an ellipsoid of revolution the planes of circular section are perpendicular to the axis of revolution of the surface; prove the same property for the hyperboloids of revolution of one and of two sheets.

10. Derive the equations of the planes of circular section through an arbitrary point for the hyperboloid of one sheet, and also for the hyperboloid of two sheets with respect to a system of axes which are parallel to the principal directions of the surfaces.

11. Solve the corresponding problem for the proper cone and for the elliptic cylinder.

12. Determine the condition under which the two planes of circular section of the elliptic cylinder which pass through a fixed point are perpendicular to each other.

112. Tangent Planes Parallel to a Given Plane. The Umbilics of a Quadric Surface. It may happen that, even though the planes of circular section of a quadric are real, yet the sections themselves fail to be real because the plane does not meet the surface in real points; a limiting case arises when a plane of circular section is tangent to the surface. In that case, if we are dealing with a family of real planes of circular sections, which are non-degenerate, the circle of section reduces to a point; such a point on a surface is called an umbilical point, or an umbilic.

DEFINITION I. **An umbilic of a quadric surface is a point on the surface at which the tangent plane is parallel to a plane through the origin which cuts the surface in a non-degenerate circle.**

Remark. It follows from this definition and from Sections 110 and 111 that umbilical points can exist at most on the central quadrics, the cone, the elliptic paraboloid and the elliptic cylinder.

Since these quadrics have at most two sets of parallel planes of circular section, the existence of umbilical points depends upon the existence of points on the surface at which the tangent plane is parallel to the planes of these sets. On account of the intrinsic interest of the question we shall preface the further discussion of umbilics by a treatment of the general question of determining points on a quadric surface at which the tangent plane is parallel to a given plane; and we shall discuss this problem for all real non-degenerate quadric surfaces.

Let $ax + by + cz = 0$ be an arbitrary plane through the origin (a, b, and c not all zero). Does there exist a point $P(\alpha, \beta, \gamma)$ on the surface Q such that the plane tangent to the surface at P is parallel to the given plane? Since the tangent plane to the surface at P is represented by the equation

$$(x - \alpha)Q_1(\alpha, \beta, \gamma) + (y - \beta)Q_2(\alpha, \beta, \gamma) + (z - \gamma)Q_3(\alpha, \beta, \gamma) = 0,$$

the conditions of the problem require that there shall exist a non-zero factor of proportionality 2ρ, such that the coördinates of P satisfy the equations

$$Q_1(\alpha, \beta, \gamma) = 2\rho a, \quad Q_2(\alpha, \beta, \gamma) = 2\rho b, \quad Q_3(\alpha, \beta, \gamma) = 2\rho c;$$

and moreover these coördinates must satisfy the condition $Q(\alpha, \beta, \gamma) = 0$. The latter equation may be written in the form

$$\alpha Q_1(\alpha, \beta, \gamma) + \beta Q_2(\alpha, \beta, \gamma) + \gamma Q_3(\alpha, \beta, \gamma) + Q_4(\alpha, \beta, \gamma) = 0,$$

which, by use of the first three equations, may be replaced by the equation

$$Q_4(\alpha, \beta, \gamma) + 2\,\rho(a\alpha + b\beta + c\gamma) = 0;$$

and this equation has the advantage of being linear in α, β, and γ. We find therefore that α, β, and γ must satisfy the following four linear equations:

$$(1)\qquad \begin{aligned} a_{11}\alpha + a_{12}\beta + a_{13}\gamma + a_{14} - \rho a &= 0,\\ a_{12}\alpha + a_{22}\beta + a_{23}\gamma + a_{24} - \rho b &= 0,\\ a_{13}\alpha + a_{23}\beta + a_{33}\gamma + a_{34} - \rho c &= 0,\\ (a_{14} + \rho a)\alpha + (a_{24} + \rho b)\beta + (a_{34} + \rho c)\gamma + a_{44} &= 0. \end{aligned}$$

If these equations are looked upon as forming a system of linear equations in α, β, γ, it follows from Corollary 2 of Theorem 24, Chapter IV (Section 54, page 102) that they possess no solution, unless the rank of the augmented matrix is less than 4. We are led therefore to the following equation for ρ:

$$R(\rho) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} - \rho a \\ a_{12} & a_{22} & a_{23} & a_{24} - \rho b \\ a_{13} & a_{23} & a_{33} & a_{34} - \rho c \\ a_{14} + \rho a & a_{24} + \rho b & a_{34} + \rho c & a_{44} \end{vmatrix} = 0.$$

We write this determinant as the sum of two determinants, using a_{14}, a_{24}, a_{34}, a_{44} as the elements of the 4th column in one and $-\rho a$, $-\rho b$, $-\rho c$, 0 as the elements of the 4th column in the other; each of these determinants is again written as the sum of two determinants by making a similar distribution of the elements of the 4th row. The equation will then take the following form:

$$\Delta + \rho \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a & b & c & 0 \end{vmatrix} - \rho \begin{vmatrix} a_{11} & a_{12} & a_{13} & a \\ a_{12} & a_{22} & a_{23} & b \\ a_{13} & a_{23} & a_{33} & c \\ a_{14} & a_{24} & a_{34} & 0 \end{vmatrix}$$

$$- \rho^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} & a \\ a_{12} & a_{22} & a_{23} & b \\ a_{13} & a_{23} & a_{33} & c \\ a & b & c & 0 \end{vmatrix} = 0.$$

Now it should be an easy matter to see that the two middle terms are equal numerically and opposite in sign; moreover the coeffi-

cient of ρ^2 is of the same form as the determinant which we denoted by the symbol $A_3(Q)$ in Section 84 (see page 172), and will by analogous notation be designated by $A_3(a, b, c)$. The equation for ρ can then be written in the simple form

$$(2) \quad \Delta = \rho^2 \times A_3(a, b, c).$$

To each root of this equation there corresponds a single value for each of the variables α, β, γ, to be determined from the equations (1), provided the rank of the coefficient matrix of these equations is 3. The discussion of the different possibilities, and also of the cases in which the roots of the equation (2) are real and distinct, real and equal, or complex is made most conveniently after the equation of the quadric surface has been reduced to the standard forms, discussed in Chapter VIII. The translation and rotation of axes which are involved in this reduction will of course affect the equation $ax + by + cz = 0$ of the plane. It is therefore of importance to establish first the following theorem.

THEOREM 5. **The value of the determinant $A_3(a, b, c)$ and the rank of its matrix are invariants of the configuration consisting of the surface Q and the plane $ax + by + cz = 0$ with respect to translation and rotation of axes.**

Proof. The most general transformation of coördinates which can be made by rotation of axes is carried out by means of the equations

$$x = \lambda_1 x_1 + \lambda_2 y_1 + \lambda_3 z_1, \; y = \mu_1 x_1 + \mu_2 y_1 + \mu_3 z_1, \; z = \nu_1 x_1 + \nu_2 y_1 + \nu_3 z_1$$

(compare Theorem 5, Chapter V, Section 63, page 121). If these expressions are substituted for x, y, and z in the equation $ax + by + cz = 0$, it is carried over into the equation $a'x_1 + b'y_1 + c'z_1 = 0$, where

$$a' = a\lambda_1 + b\mu_1 + c\nu_1, \quad b' = a\lambda_2 + b\mu_2 + c\nu_2, \quad c' = a\lambda_3 + b\mu_3 + c\nu_3.$$

We observe now that this transformation of the coefficients of x, y, z in the equation of the plane is exactly the same as the transformation of the coefficients a_{14}, a_{24}, a_{34} of the linear terms in Q under rotation of axes (compare page 212); consequently the determinant $A_3(a, b, c)$ is transformed by rotation of axes exactly

like the discriminant of the quadric surface $q(x, y, z) + 2\,ax + 2\,by + 2\,cz = 0$. It follows therefore from Theorems 4 and 5, Chapter VIII (Section 94, pages 203 and 205) that the value of the determinant $A_3(a, b, c)$ and the rank of its matrix are invariant with respect to rotation of axes. That this invariance also holds with respect to translation of axes becomes evident if we recall that the coefficients of the second degree terms in Q are not changed by translation of axes (compare proof of Theorem 1, Chapter VIII, Section 93, page 199) and if we observe that the transformation

$$x = x' + p, \quad y = y' + q, \quad z = z' + r$$

carries the equation $ax + by + cz = 0$ over into the function $ax' + by' + cz' + ap + bq + cr = 0$, so that the coefficients a, b, and c are also invariant under translation of axes.

In the further discussion of our problem we shall now be able to use the standard forms of the quadric surfaces.

CASE I. $r_4 = 4, \quad r_3 = 3.$

(*a*) *Ellipsoid.* The standard form of the equation is

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1.$$

We find that $\Delta = -\dfrac{1}{p^2q^2r^2}$, and $A_3(a, b, c) = \begin{vmatrix} \frac{1}{p^2} & 0 & 0 & a \\ 0 & \frac{1}{q^2} & 0 & b \\ 0 & 0 & \frac{1}{r^2} & c \\ a & b & c & 0 \end{vmatrix} =$ $-\left(\dfrac{a^2}{q^2r^2} + \dfrac{b^2}{r^2p^2} + \dfrac{c^2}{p^2q^2}\right)$. The equation (2) becomes therefore $\rho^2(a^2p^2 + b^2q^2 + c^2r^2) = 1$. It has two real roots for every set of values of a, b, and c and, since $r_3 = 3$, a single set of values of α, β, γ is given by equations (1) for each root of (2). Therefore, to every plane there correspond two points on the ellipsoid at which the tangent planes are parallel to the given plane.

(*b*) *Hyperboloid of One Sheet.* From the standard form of the equation, namely,

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1,$$

we find $\Delta = \frac{1}{p^2q^2r^2}$, and $A_3(a, b, c) = \frac{a^2}{q^2r^2} + \frac{b^2}{r^2p^2} - \frac{c^2}{p^2q^2}$. In this case $\Delta > 0$, although $A_3(a, b, c)$ is positive, zero or negative according as $a^2p^2 + b^2q^2 - c^2r^2$ is positive, zero or negative. Since we have again $r_3 = 3$, we conclude that if a, b, and c are so chosen that $a^2p^2 + b^2q^2 > c^2r^2$, there are two points on the surface at which the tangent plane is parallel to the plane $ax + by + cz = 0$; if $a^2p^2 + b^2q^2 < c^2r^2$, there are no such points on the surface, and if $a^2p^2 + b^2q^2 = c^2r^2$, there is no finite point on the surface which has this property.

(c) *Hyperboloid of Two Sheets.* Using the standard form $\frac{x^2}{p^2} - \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$, we find $\Delta = -\frac{1}{p^2q^2r^2}$ and $A_3(a, b, c) = -\frac{a^2}{q^2r^2} + \frac{b^2}{r^2p^2} + \frac{c^2}{p^2q^2}$. Now $\Delta < 0$, so that equation (2) furnishes real values of ρ only in case $b^2q^2 + c^2r^2 < a^2p^2$. In this case there will be two real points on the surface of the desired kind; in no other case will points of this kind exist at finite distance.

The conclusions for this case are therefore as follows:

Theorem 6. **On the ellipsoid there are, for every plane in space, two finite points at which the tangent plane is parallel to the given plane; on the hyperboloid of one or two sheets two such points exist for certain planes but none for others.**

Case II. $r_4 = 4$, $r_3 = 2$.

(a) *Elliptic Paraboloid.* The standard form of the equation is

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 2\,nz.$$

We find $\Delta = -\frac{n^2}{p^2q^2}$ and $A_3(a, b, c) = -\frac{c^2}{p^2q^2}$. Equation (2) becomes therefore $n^2 = \rho^2c^2$. If $c = 0$, there is no finite value of ρ which satisfies this equation and hence no finite point on the surface which satisfies the conditions of our problem. If $c \neq 0$, we have $\rho = \pm\frac{n}{c}$; let $\rho_1 = \frac{n}{c}$ and $\rho_2 = -\frac{n}{c}$. If we substitute ρ_1 or ρ_2 in the equations (1) for this case, we obtain a system of linear equations whose augmented matrix is

$$\left\|\begin{matrix} \frac{1}{p^2} & 0 & 0 & -\rho_1 a \\ 0 & \frac{1}{q^2} & 0 & -\rho_1 b \\ 0 & 0 & 0 & -2n \\ \rho_1 a & \rho_1 b & 0 & 0 \end{matrix}\right\| \quad \text{or} \quad \left\|\begin{matrix} \frac{1}{p^2} & 0 & 0 & -\rho_2 a \\ 0 & \frac{1}{q^2} & 0 & -\rho_2 b \\ 0 & 0 & 0 & 0 \\ \rho_2 a & \rho_2 b & -2n & 0 \end{matrix}\right\|.$$

The rank of each of these matrices is clearly less than 4. The three-rowed principal minors formed from the 1st, 2nd, and 4th rows and columns have the values $\rho_1^2\left(\frac{a^2}{q^2}+\frac{b^2}{p^2}\right)$ and $\rho_2^2\left(\frac{a^2}{q^2}+\frac{b^2}{p^2}\right)$ respectively; and since ρ_1 and ρ_2 are both different from zero, we conclude that these two matrices are both of rank 3.

The rank of the coefficient matrix for the first of these systems of equations is manifestly less than 3; for the second system the coefficient matrix contains the non-vanishing three-rowed minor

$$\begin{vmatrix} \frac{1}{p^2} & 0 & 0 \\ 0 & \frac{1}{q^2} & 0 \\ \rho_2 a & \rho_2 b & -2n \end{vmatrix}$$

whose value is $-\frac{2n}{p^2q^2}$. In accordance with Corollary 1 of Theorem 24, Chapter IV (Section 54, page 102) we conclude that for $\rho = \rho_1$, the system (1) has no solutions, while for $\rho = \rho_2$ it possesses one solution. For $\rho = \rho_2$, the equations (1) are $\frac{\alpha}{p^2} - \rho_2 a = 0$, $\frac{\beta}{q^2} - \rho_2 b = 0$, $\rho_2(a\alpha + b\beta) - 2n\gamma = 0$. From these we obtain, since $\rho_2 = -\frac{n}{c}$, the following solution of our problem:

$$\alpha = -\frac{anp^2}{c}, \quad \beta = -\frac{bnq^2}{c}, \quad \gamma = \frac{n(a^2p^2 + b^2q^2)}{2c^2}.$$

Thus, while the equation (2) has two real solutions in this case, only one of them gives rise to a point (α, β, γ) on the surface at which the tangent plane is parallel to an arbitrarily given plane through the origin, except when this plane has the equation $ax + by = 0$.

(*b*) *Hyperbolic Paraboloid.* From the standard form of the equation

$$\frac{x^2}{p^2} - \frac{y^2}{q^2} = 2\,nz$$

we obtain $\Delta = \dfrac{n^2}{p^2q^2}$ and $A_3(a, b, c) = \dfrac{c^2}{p^2q^2}$. The equation (2) reduces, as in (*a*) to the form $n^2 = \rho^2c^2$. For $c = 0$, there is no finite solution of the problem; for $c \neq 0$, we find as before, $\rho_1 = \dfrac{n}{c}$ and $\rho_2 = -\dfrac{n}{c}$. For these two values of ρ the augmented matrices of the systems of equations (1) become

$$\begin{Vmatrix} \frac{1}{p^2} & 0 & 0 & -\rho_1 a \\ 0 & -\frac{1}{q^2} & 0 & -\rho_1 b \\ 0 & 0 & 0 & -2\,n \\ \rho_1 a & \rho_1 b & 0 & 0 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} \frac{1}{p^2} & 0 & 0 & -\rho_2 a \\ 0 & -\frac{1}{q^2} & 0 & -\rho_2 b \\ 0 & 0 & 0 & 0 \\ \rho_2 a & \rho_2 b & -2\,n & 0 \end{Vmatrix}.$$

It is seen that, as in (*a*), both these matrices are of rank 3; also that the rank of the coefficient matrix for the first system of equations is 2 and the rank of the coefficient matrix for the second system of equations is 3. The conclusion is therefore the same as for the elliptic paraboloid; we obtain the point $\alpha = -\dfrac{anp^2}{c}$, $\beta = \dfrac{bnq^2}{c}$, $\gamma = \dfrac{n(a^2p^2 - b^2q^2)}{2\,c^2}$.

Theorem 7. **On the elliptic paraboloid and on the hyperbolic paraboloid, there is for every plane in space, except for planes parallel to the axis of the surface, one point at which the tangent plane is parallel to the given plane.**

Case III. $r_4 = r_3 = 3$.

The only real quadric in this case is the proper cone, whose standard equation may be put in the form

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 0.$$

We find $\Delta = 0$ and

$$A_3(a, b, c) = \begin{vmatrix} \frac{1}{p^2} & 0 & 0 & a \\ 0 & \frac{1}{q^2} & 0 & b \\ 0 & 0 & -\frac{1}{r^2} & c \\ a & b & c & 0 \end{vmatrix} = \frac{p^2a^2 + b^2q^2 - c^2r^2}{p^2q^2r^2}.$$

If $A_3(a, b, c) \neq 0$, the only solution of equation (2) is $\rho = 0$, and equations (1) reduce to the equations for the vertex (compare Theorem 13, Chapter VII, Section 85, page 177). Since at the vertex $Q_1 = Q_2 = Q_3 = 0$, the tangent plane at this point does not exist and our problem has therefore no solution in this case. On the other hand, if $A_3(a, b, c) = 0$, i.e., if $p^2a^2 + b^2q^2 - c^2r^2 = 0$, every value of ρ satisfies equation (2). The augmented matrix of the system of equations (1) is $\left\| \begin{matrix} \frac{1}{p^2} & 0 & 0 & -\rho a \\ 0 & \frac{1}{q^2} & 0 & -\rho b \\ 0 & 0 & -\frac{1}{r^2} & -\rho c \\ \rho a & \rho b & \rho c & 0 \end{matrix} \right\|$; its determinant is equal to $-\rho^2 A_3(a, b, c)$ and vanishes. It should be clear that the rank of this matrix is 3 and that the rank of the coefficient matrix of the system of equations (1) is also 3. Hence, for every value of ρ there is one point on the surface which satisfies the conditions of our problem. By solving the system (1), we find

$$\alpha = \rho a p^2, \quad \beta = \rho b q^2, \quad \gamma = -\rho c r^2.$$

As ρ varies, these equations are the parametric equations of a line and it is readily seen that this line lies entirely on the cone. For, since $a^2p^2 + b^2q^2 - c^2r^2 = 0$, it follows that $\frac{\alpha^2}{p^2} + \frac{\beta^2}{q^2} - \frac{\gamma^2}{r^2} = \rho^2(a^2p^2 + b^2q^2 - c^2r^2) = 0$, independently of the value of ρ. And the tangent plane to the cone at any point on this line is represented by the equation $\frac{\alpha x}{p^2} + \frac{\beta y}{q^2} - \frac{\gamma z}{r^2} = 0$, i.e. by the equation

$ax + by + cz = 0$. We have therefore reached the following conclusion:

THEOREM 8. **If and only if a, b, and c are so chosen that $a^2p^2 + b^2q^2 - c^2r^2 = 0$, the plane $ax + by + cz = 0$ will be tangent to the cone $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 0$; this plane touches the surface along the line on the cone whose parametric equations are $x = \rho a p^2$, $y = \rho b q^2$, $z = -\rho c r^2$, ρ being the parameter. For any other choice of a, b, and c there will be no point on the cone at which the tangent plane is parallel to the plane $ax + by + cz = 0$.**

CASE IV. $r_4 = 3$, $r_3 = 2$.

(a) *Elliptic Cylinder.* The standard equation is $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$; $\Delta = 0$ and $A_3(a, b, c) = -\frac{c^2}{p^2q^2}$. If $c \neq 0$, the only solution of equation (2) is $\rho = 0$; the system (1) is inconsistent for this value of ρ and we have therefore no solution of the problem. If $c = 0$, the system of equations (1) reduces to

$$\frac{\alpha}{p^2} = \rho a, \quad \frac{\beta}{q^2} = \rho b, \quad \rho(a\alpha + b\beta) = 1.$$

Values of ρ can always be determined for which this system is consistent; with these values of ρ, we find for α and β the following results:

$$\alpha = \pm \frac{ap^2}{\sqrt{a^2p^2 + b^2q^2}}, \quad \beta = \pm \frac{bq^2}{\sqrt{a^2p^2 + b^2q^2}}.$$

These equations determine a line parallel to the Z-axis, which lies entirely on the cylinder; and the plane tangent to the cylinder at any point on this line is represented by the equation $ax + by = \pm\sqrt{a^2p^2 + b^2q^2}$.

(b) *Hyperbolic Cylinder.* From the standard equation $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$, we find $\Delta = 0$ and $A_3(a, b, c) = \frac{c^2}{p^2q^2}$. Conditions are similar to those in (a). If $c \neq 0$, there is no tangent plane parallel to the plane $ax + by + cz = 0$. If $c = 0$, the values of ρ and of α, β, γ are to be determined from the equations (1) which reduce in this case to

$$\alpha = \rho a p^2, \quad \beta = -\rho b q^2, \quad \rho(a\alpha + b\beta) = 1.$$

Elimination of α and β leads to the equation $\rho^2(a^2p^2 - b^2q^2) = 1$; hence if and only if $a^2p^2 - b^2q^2 > 0$, does there exist a real point which satisfies the conditions of the problem. In this case, we find

$$\alpha = \pm\frac{ap^2}{\sqrt{a^2p^2 - b^2q^2}}, \quad \beta = \mp\frac{bq^2}{\sqrt{a^2p^2 - b^2q^2}}.$$

The line determined by these equations lies entirely on the surface, and the equation of the plane tangent to the cylinder at any point of this line is $ax + by = \pm\sqrt{a^2p^2 - b^2q^2}$.

CASE V. $r_4 = 3$, $r_3 = 1$.

In this case the locus of the equation $Q = 0$ is a parabolic cylinder; the equation of this surface may be reduced to the standard form $y^2 - 4\,px = 0$. We have

$$\Delta = 0 \text{ and } A_3(a, b, c) = \begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & c \\ a & b & c & 0 \end{vmatrix} = 0.$$

Hence equation (2) imposes no restriction on the choice of ρ. The system of equations (1) takes the form

$$-2\,p - \rho a = 0, \quad \beta - \rho b = 0, \quad -\rho c = 0, \quad (-2\,p + \rho a)\alpha + \rho b\beta + \rho c\gamma = 0.$$

If $c \neq 0$, the third equation requires that $\rho = 0$, which leads to a contradiction with the first equation, since $p \neq 0$; in this case there is therefore no solution. For a similar reason, we must have $a \neq 0$; and in this case we find $\rho = -\dfrac{2\,p}{a}$, and hence $\beta = \rho b = -\dfrac{2\,pb}{a}$, $\alpha = \dfrac{pb^2}{a^2}$. These equations represent a line which lies entirely on the cylinder; and the tangent plane to the surface at any point on this line is given by the equation $y\beta - 2\,p(x + \alpha) = 0$, that is, by the equation $ax + by + \dfrac{pb^2}{a} = 0$. There is no finite point on the cylinder at which the tangent plane is parallel to the plane $ax + by = 0$, if $a = 0$; that is, there is no tangent plane parallel to the plane $y = 0$.

We summarize now the conclusions reached in Cases IV and V in the following theorem.

THEOREM 9. **The tangent planes to the elliptic cylinder, the hyperbolic cylinder and the parabolic cylinder are all parallel to the generating line of the cylinder; they have contact with the surface along an entire generator. The elliptic cylinder $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$ has two tangent planes parallel to an arbitrary plane $ax + by = 0$ through its axis, namely, the planes $ax + by = \pm\sqrt{a^2p^2 + b^2q^2}$. The hyperbolic cylinder $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$ has two tangent planes parallel to an arbitrary plane $ax + by = 0$ through its axis, namely, the planes $ax + by = \pm\sqrt{a^2p^2 - b^2q^2}$, unless $a^2p^2 - b^2q^2 \leqq 0$, in which case no such plane exists. The parabolic cylinder $y^2 = 4px$ has one tangent plane parallel to the arbitrary plane $ax + by = 0$ through the Z-axis, namely, the plane $ax + by + \frac{pb^2}{a} = 0$, if $a \neq 0$; if $a = 0$ no such plane exists.**

113. The Umbilics of a Quadric Surface, *continued.* The determination of the umbilics on the central quadrics, the cone, the elliptic paraboloid and the elliptic cylinder can now be effected, on the basis of Definition I (Section 112, page 253), by combining the results of Theorems 6, 7, 8, and 9 with those of Theorem 4. We shall take the equations of these surfaces in the standard forms used in the preceding section.

(*a*) *Ellipsoid.*

If $p < q < r$, the equations of the planes of circular section, through the origin, of the ellipsoid $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} - 1 = 0$ are (see Section 110):

$$r\sqrt{q^2 - p^2}\,x \pm p\sqrt{r^2 - q^2}\,z = 0.$$

Hence, in the notation of Section 112, we have $a = r\sqrt{q^2 - p^2}$, $b = 0$, $c = \pm p\sqrt{r^2 - q^2}$, and $\rho = \pm\frac{1}{\sqrt{a^2p^2 + b^2q^2 + c^2r^2}} = \pm\frac{1}{pr\sqrt{r^2 - p^2}}$, the double sign of ρ being independent of that of c. The first three equations of the system (1) of Section 112 are for this case $\frac{\alpha}{p^2} = \rho a$, $\frac{\beta}{q^2} = \rho b$, $\frac{\gamma}{r^2} = \rho c$. We find therefore

$$\alpha = \rho a p^2 = \pm\frac{p\sqrt{q^2 - p^2}}{\sqrt{r^2 - p^2}}, \quad \beta = 0, \quad \gamma = \rho c r^2 = \pm\frac{r\sqrt{r^2 - q^2}}{\sqrt{r^2 - p^2}},$$

the double signs of α and γ being independent of each other. The conclusion is given by the following theorem.

THEOREM 10. **The ellipsoid $\frac{x^2}{p^2}+\frac{y^2}{q^2}+\frac{z^2}{r^2}=1$, in which $p<q<r$, has four umbilics. Their coördinates are $\left(\pm\frac{p\sqrt{q^2-p^2}}{\sqrt{r^2-p^2}},\ 0,\ \pm\frac{r\sqrt{r^2-q^2}}{\sqrt{r^2-p^2}}\right)$; they lie on the ellipse in which the surface is cut by the XZ-plane.**

(*b*) *Hyperboloid of One Sheet.* If the equation be taken in the standard form $\frac{x^2}{p^2}+\frac{y^2}{q^2}-\frac{z^2}{r^2}=0$, $p<q$, the central planes of circular section are (compare Exercise 10, Section 111, page 252):

$$r\sqrt{q^2-p^2}\,x \mp p\sqrt{q^2+r^2}\,z = 0.$$

Hence $a = r\sqrt{q^2-p^2}$, $b=0$, $c=\mp p\sqrt{q^2+r^2}$; and $a^2p^2 + b^2q^2 - c^2r^2 = p^2r^2(q^2-p^2) - r^2p^2(q^2+r^2) = -p^2r^2(p^2+r^2) < 0$. Therefore, the hyperboloid of one sheet has no umbilics.

(*c*) *Hyperboloid of Two Sheets.* We proceed as in the preceding case, for the equation $\frac{x^2}{p^2}-\frac{y^2}{q^2}-\frac{z^2}{r^2}=1$, $q<r$. The central planes of circular section are represented by the equations

$$q\sqrt{r^2+p^2}\,x \pm p\sqrt{r^2-q^2}\,y = 0.$$

We have $a=q\sqrt{r^2+p^2}$, $b=\pm p\sqrt{r^2-q^2}$, $c=0$; $a^2p^2-b^2q^2-c^2r^2 = p^2q^2(r^2+p^2)-q^2p^2(r^2-q^2) = p^2q^2(p^2+q^2) > 0$. Therefore there are umbilical points on this surface. Their actual determination proceeds as in the case of the ellipsoid; the reader should have no difficulty in completing the proof of the following theorem:

THEOREM 11. **On the hyperboloid of one sheet there are no umbilics. On the hyperboloid of two sheets, which is not a surface of revolution, there are four umbilics; if the equation of the surface is $\frac{x^2}{p^2}-\frac{y^2}{q^2}-\frac{z^2}{r^2}=1$, $q<r$, their coördinates are $\left(\pm\frac{p\sqrt{r^2+p^2}}{\sqrt{p^2+q^2}},\ \pm\frac{q\sqrt{r^2-q^2}}{\sqrt{p^2+q^2}},\ 0\right)$, the two double signs being independent of each other.**

(*d*) *Cone.* If the equation be written in the form $\frac{x^2}{p^2}+\frac{y^2}{q^2}-\frac{z^2}{r^2}=0$, $p<q$, the planes of circular section through the origin are:

$$r\sqrt{q^2-p^2}\,x \pm p\sqrt{q^2+r^2}\,z = 0.$$

Here $a = r\sqrt{q^2 - p^2}$, $b = 0$, $c = \pm p\sqrt{q^2 + r^2}$; and $a^2p^2 + b^2q^2 - c^2r^2 = p^2r^2(q^2 - p^2) - p^2r^2(q^2 + r^2) = - p^2r^2(p^2 + r^2) \neq 0$. Therefore the cone has no umbilics.

(*e*) *Elliptic Paraboloid.* From the standard equation $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 2\,nz$, $p < q$, we obtain for the planes of circular section through the origin, the equations $\sqrt{q^2 - p^2}\, x \pm pz = 0$. Hence

$$a = \sqrt{q^2 - p^2}, \quad b = 0, \quad c = \pm p.$$

There are therefore two umbilics on this surface; from the formulas, given in Section 112, Case II, (*a*), page 258, we find that their coördinates are

$$\alpha = \pm\, pn\sqrt{q^2 - p^2}, \quad \beta = 0, \quad \gamma = \frac{n(q^2 - p^2)}{2}.$$

(*f*) *Elliptic Cylinder.* We take the equation in the form $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$, $p < q$, and we find that the circular sections through the origin lie in the planes represented by the equations $\sqrt{q^2 - p^2}\, x \pm pz = 0$. Since these planes are not parallel to the generators of the cylinder, it follows from Theorem 9 that there are no umbilics on this surface.

The results found in cases (*d*), (*e*), and (*f*) lead to the following theorem.

THEOREM 12. **There are no umbilics on the cone, nor on the elliptic cylinder. There are two umbilics on the elliptic paraboloid which is not a surface of revolution; if the equation of this surface be reduced to the form $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 2\,nz$, $p < q$, the coördinates of the umbilics are $\left(\pm pn\sqrt{q^2 - p^2}, 0, \frac{n(q^2 - p^2)}{2}\right)$.**

(*g*) *Surfaces of Revolution.* We have seen in Theorem 4 that for a surface of revolution there is one plane of circular section through every point of space; these surfaces can therefore have at most two umbilics. The single central plane of circular section may in these cases be obtained from the results already found, by setting two of the discriminating numbers equal to each other. So, for example, if in the ellipsoid, treated under (*a*), we put $p = q$, the four umbilics reduce to two, namely to the points $(0, 0, \pm r)$, that

is, to the points in which the axis of revolution meets the surface. The reader should have no difficulty in obtaining the corresponding result for the hyperboloid of revolution of two sheets and for the paraboloid of revolution. On a sphere every point is an umbilic.

THEOREM 13. **On the ellipsoid of revolution, the hyperboloid of revolution of two sheets and on the paraboloid of revolution, the umbilics are the points in which the surface is met by the axis of revolution; on the other quadrics of revolution there are no umbilics.**

We summarize the results of this section as follows:

Umbilics exist on the ellipsoid, the hyperboloid of two sheets and the elliptic paraboloid. If these surfaces are not surfaces of revolution, the number of umbilics on them are 4, 4, and 2 respectively; if they are surfaces of revolution, the umbilics fall two by two into the points where the surface is met by the axis of revolution.

114. Exercises.

1. Determine the planes parallel to the plane $x - 2y + z = 0$ which are tangent to the following surfaces:

(a) $x^2 - 6y^2 - 3z^2 = 1$, $\quad(b)$ $2x^2 + 4y^2 + 5z^2 = 1$,
(c) $x^2 + 4y^2 = 2z$, $\quad(d)$ $4x^2 - y^2 = 2z$.

2. Determine the umbilics on each of the following surfaces:

(a) $\dfrac{x^2}{4} - \dfrac{y^2}{9} - z^2 = 1$, $\quad(b)$ $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$,

(c) $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 2z$, $\quad(d)$ $x^2 + y^2 + 4z^2 = 12$.

3. Prove that the planes through the point $P(\alpha, \beta, \gamma)$ which are parallel to the planes tangent to the cone $\dfrac{x^2}{p^2} - \dfrac{y^2}{q^2} + \dfrac{z^2}{r^2} = 0$ are tangent to a cone whose vertex is at P.

4. If u, v, w are called the "coördinates" of the plane $ux + vy + wz + 1 = 0$, set up the equation which the coördinates of a plane must satisfy in order that the plane be tangent to the cone $\dfrac{(x-\alpha)^2}{p^2} + \dfrac{(y-\beta)^2}{q^2} - \dfrac{(z-\gamma)^2}{r^2} = 0$.

5. Determine the umbilics on the surface $-\dfrac{x^2}{p^2} + \dfrac{y^2}{q^2} - \dfrac{z^2}{r^2} = 1$, $p < r$.

6. Prove that the planes of circular section of the quadric surface $Q(x, y, z) = 0$ are also planes of circular section of all the quadric surfaces whose equations have the form $Q(x, y, z) + k(x^2 + y^2 + z^2) = 0$.

7. Prove that the four umbilics of an ellipsoid are the vertices of a rectangle; and determine the condition under which they will be the vertices of a square.

8. Prove that the umbilics of the ellipsoids with the same center and the

same principal directions, whose semi-axes are kp, kq, and kr, in which p, q, and r are fixed while k is variable, lie on four lines through the common center; and determine the direction cosines of these lines.

9. Determine the condition under which the tangent planes to an elliptic paraboloid at the umbilical points are perpendicular to each other.

10. Prove that the planes of circular section of an hyperboloid of one sheet are also planes of circular section of its asymptotic cone.

11. Prove that the tangent planes to an hyperboloid of two sheets at its umbilical points are planes of circular section of its asymptotic cone.

12. Prove that the planes of circular section of a proper cone cut a tangent plane of the cone in lines which make equal angles with the generator of the cone along which the tangent plane touches it.

CHAPTER X

PROPERTIES OF CENTRAL QUADRIC SURFACES

115. Conjugate Diameters and Conjugate Diametral Planes of Central Quadrics. Enveloping Cylinder.

DEFINITION I. **A *diameter of a central quadric surface* is a chord which passes through the center of the surface.**

The common form of the standard equations of the central quadrics is

$$(1)\quad m_1x^2 + m_2y^2 + m_3z^2 = 1.$$

Here m_1, m_2, and m_3 are the quotients of the discriminating numbers of the surface by $-\dfrac{\Delta}{A_{44}}$; the center of the surface is at the origin and the principal directions are the directions of the coördinate axes.

Let us now consider an arbitrary diameter d_1 of the surface; we shall designate its direction cosines by λ_1, μ_1, ν_1. The diametral plane corresponding to this direction (see Definition X, Chapter VII and Theorem 17, Section 88, page 186) in the surface (1) is given by the equation

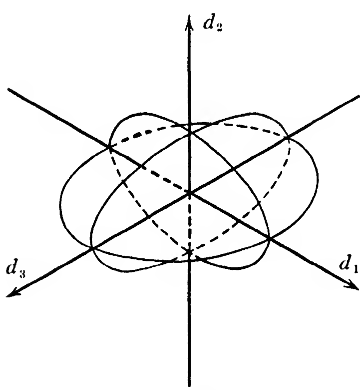

FIG. 35

$$(2)\quad m_1\lambda_1x + m_2\mu_1y + m_3\nu_1z = 0.$$

If a second diameter d_2, with direction cosines λ_2, μ_2, ν_2, lies in this plane,

$$(3)\quad m_1\lambda_1\lambda_2 + m_2\mu_1\mu_2 + m_3\nu_1\nu_2 = 0,$$

(compare Theorem 21, Chapter IV, Section 52, page 99); and the equation of the diametral plane corresponding to the direction of d_2 has the equation

$$(4)\quad m_1\lambda_2x + m_2\mu_2y + m_3\nu_2z = 0.$$

Equation (3) can now be interpreted as stating that the line d_1 lies in the plane (4). We have therefore proved the following theorem (see Fig. 35).

THEOREM 1. **If one diameter of a central quadric surface lies in the diametral plane determined by the direction of another diameter, then the second diameter lies in the diametral plane determined by the direction of the first.**

COROLLARY. **The plane determined by two diameters is the diametral plane which corresponds to the direction of the line of intersection of the diametral planes of the first two diameters.**

Proof. The diameter d_3 in which the planes (2) and (4) intersect has direction cosines λ_3, μ_3, ν_3 determined by the proportion

$$(5)\quad \lambda_3 : \mu_3 : \nu_3 = m_2m_3\begin{vmatrix} \mu_1 & \nu_1 \\ \mu_2 & \nu_2 \end{vmatrix} : m_3m_1\begin{vmatrix} \nu_1 & \lambda_1 \\ \nu_2 & \lambda_2 \end{vmatrix} : m_1m_2\begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{vmatrix}.$$

The diametral plane of this line* is represented by the equation

$$m_1m_2m_3\begin{vmatrix} \mu_1 & \nu_1 \\ \mu_2 & \nu_2 \end{vmatrix} x + m_2m_3m_1\begin{vmatrix} \nu_1 & \lambda_1 \\ \nu_2 & \lambda_2 \end{vmatrix} y + m_3m_1m_2\begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{vmatrix} z = 0.$$

But this equation is equivalent to the equation $\begin{vmatrix} x & y & z \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \end{vmatrix} = 0,$ which is indeed the equation of the plane determined by the lines d_1 and d_2.

The corollary is also proved by the fact, that since d_3 lies in the diametral planes of d_1 and d_2, the diametral plane of d_3 contains the diameters d_1 and d_2.

Of the three diameters d_1, d_2, and d_3 any two determine the diametral plane of the third; and of the three planes (d_1, d_2), (d_2, d_3), and (d_3, d_1) any one is the diametral plane of the line of intersection of the other two.

The correspondence between diameters and diametral planes is a reciprocal one-to-one correspondence: not only is there one and only one diametral plane of any line through the center, but there is also one and only one diameter of which any given plane through

* Whenever it can be done without danger of confusion, we shall use the phrase "diametral plane of a line l" in place of the more exact but less convenient expression "diametral plane corresponding to the direction of the line l."

the center is the diametral plane. For if $ax + by + cz = 0$ is an arbitrary plane through the origin, there is one and only one set of values λ, μ, ν such that $m_1\lambda : m_2\mu : m_3\nu = a : b : c$, hence one diameter d whose diametral plane $m_1\lambda x + m_2\mu y + m_3\nu z = 0$ coincides with the given plane.

We introduce now the following definition.

Definition II. **A set of three diameters of a central quadric, such that the diametral plane of any one of them is the plane determined by the other two, is called a set of *conjugate diameters;* and a set of three diametral planes, such that any one of them is the diametral plane of the line of intersection of the other two, is called a set of *conjugate diametral planes.***

We shall find it convenient to refer to such sets as a **conjugate set of diameters** and a **conjugate set of diametral planes.**

In terms of this definition we have the following theorem.

Theorem 2. **For every diameter (diametral plane) of a central quadric there exist an infinite number of conjugate sets; for every pair of diameters (diametral planes), of which one lies in the diametral plane of the other (passes through the diameter of the other) there exists one conjugate set. For every diameter together with a diametral plane passing through it, there exists one set of conjugate diameters and one set of diametral planes, such that the diameters of the first set are the lines of intersection of the planes of the second set.**

Remark 1. The axes of symmetry of the ellipsoid furnish an example of a conjugate set of diameters.

Remark 2. Any two of a conjugate set of diameters are "conjugate diameters" of the conic section in which their plane cuts the quadric, where the words in quotation marks are to be understood in the sense in which they are used in Plane Analytical Geometry.

The coördinates (α, β, γ) of the point P in which the diameter d of direction cosines λ, μ, ν meets the surface, are proportional to λ, μ, ν; that is, $\alpha = \lambda s$, $\beta = \mu s$, $\gamma = \nu s$. The tangent plane to the surface at this point may therefore be represented by the equation $m_1\lambda x + m_2\mu y + m_3\nu z = 1$. Comparison with equation (2) shows that this plane is parallel to the diametral plane of d.

And if $P(\alpha, \beta, \gamma)$ is a point of the conic in which the diametral plane D of the diameter d cuts the surface, then $m_1\alpha\lambda + m_2\beta\mu + m_3\gamma\nu = 0$; and the tangent plane to the surface at this point is

represented by the equation $m_1\alpha x + m_2\beta y + m_3\gamma z = 1$. Consequently this tangent plane is parallel to the line d (compare Theorem 21, Chapter IV, Section 52, page 99); hence there exists also a tangent line through P parallel to d. If P moves along the curve of intersection of the surface with the plane D, these tangent lines which are parallel to d generate a cylinder.

Definition III. **The *enveloping cylinder* of a quadric surface corresponding to a line d is the cylinder generated by the tangent lines to the surface which are parallel to d.**

We have therefore obtained moreover the following result.

Theorem 3. **The tangent planes to a quadric surface at the points where it is met by a diameter d are parallel to the diametral plane of d. The tangent planes at the points where it is met by the diametral plane of d are parallel to d; the tangent lines drawn through these points and parallel to d form an enveloping cylinder of the surface.**

116. Exercises.

1. Show that the enveloping cylinder of an ellipsoid is an elliptic cylinder for every direction of the generator. Obtain its equation.

2. Show that the enveloping cylinder of an hyperboloid of two sheets is an hyperbolic cylinder for every direction of the generator; obtain its equation.

3. Show that the enveloping cylinder of an hyperboloid of one sheet is an elliptic cylinder for some directions of the generator and an hyperbolic cylinder for other directions of the generator; obtain the equation of the enveloping cylinder.

4. Determine the directions of the generator for which the enveloping cylinder of an hyperboloid of one sheet will be an elliptic cylinder, and also the directions for which it will be an hyperbolic cylinder.

5. Determine the diameter of the surface $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$ which is conjugate to the diameters whose direction cosines are proportional to $-2 : -1 : 2$ and $1 : -9 : 4$ respectively; determine also the diametral planes of these diameters.

6. Solve the corresponding problem for the surface $x^2 + \frac{y^2}{4} + \frac{z^2}{6} = 1$ and the directions $1 : 2 : -3$ and $-2 : 3 : -1$.

7. Determine the enveloping cylinders of the ellipsoid in the preceding problem for the directions of each of the three conjugate diameters.

8. Determine for the surface of Exercise 6 the diametral plane which is conjugate to the planes $x + 2y + 2z = 0$ and $4x + y - z = 0$.

117. Conjugate Diameters of the Ellipsoid. We shall denote the length of the chord determined by the diameter d of a quadric

surface by $2\,\delta$, the points where d meets the surface by $E(\alpha, \beta, \gamma)$ and $E'(\alpha', \beta', \gamma')$, and the direction cosines of d by λ, μ, ν. If several diameters are under consideration at the same time, we shall distinguish between the numbers related to them by the use of subscripts.

The parametric equations of d may be taken in the form

$$(1)\quad x = \lambda s, \quad y = \mu s, \quad z = \nu s.$$

The number δ is the numerical value of the roots of the equation

$$(2)\quad (m_1\lambda^2 + m_2\mu^2 + m_3\nu^2)s^2 = 1,$$

provided these roots are real; and the coördinates of E and E' are obtained by substituting these roots for s in equations (1). It should be clear that the roots of equation (2) never coincide, that for the ellipsoid they are always real and finite, while for the hyperboloids of 1 or 2 sheets, they will be real and finite, infinite, or imaginary according as $m_1\lambda^2 + m_2\mu^2 + m_3\nu^2$ is positive, zero, or negative.

We will show now that the ellipsoid is the only quadric for which there exist sets of conjugate diameters, for each of which the roots of the equation (2) are real and finite. That such sets do exist for the ellipsoid follows from Theorem 2 (Section 115, page 270) in conjunction with the observation in the preceding paragraph. Suppose now that we have three diameters d_1, d_2, and d_3 such that

$$(3)\quad m_1\lambda_1^2 + m_2\mu_1^2 + m_3\nu_1^2 > 0, \qquad m_1\lambda_2^2 + m_2\mu_2^2 + m_3\nu_2^2 > 0,$$
$$m_1\lambda_3^2 + m_2\mu_3^2 + m_3\nu_3^2 > 0, \quad \text{and}$$
$$(4)\quad m_1\lambda_2\lambda_3 + m_2\mu_2\mu_3 + m_3\nu_2\nu_3 = 0, \quad m_1\lambda_3\lambda_1 + m_2\mu_3\mu_1 + m_3\nu_3\nu_1 = 0,$$
$$m_1\lambda_1\lambda_2 + m_2\mu_1\mu_2 + m_3\nu_1\nu_2 = 0.$$

From the first two of equations (4) we derive equations (5) of Section 115; and from these we conclude that there exists a non-zero constant k, such that

$$\begin{vmatrix} \mu_1 & \nu_1 \\ \mu_2 & \nu_2 \end{vmatrix} = \frac{k\lambda_3}{m_2m_3}, \quad \begin{vmatrix} \nu_1 & \lambda_1 \\ \nu_2 & \lambda_2 \end{vmatrix} = \frac{k\mu_3}{m_3m_1}, \quad \begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{vmatrix} = \frac{k\nu_3}{m_1m_2}.$$

If we multiply the inequalities (3_1) and (3_2) and subtract the square of equation (4_3) from the result, we obtain, by an easy rearrangement of terms:

$$m_1m_2(\lambda_1\mu_2 - \lambda_2\mu_1)^2 + m_2m_3(\mu_1\nu_2 - \mu_2\nu_1)^2 + m_3m_1(\nu_1\lambda_2 - \nu_2\lambda_1)^2 > 0.$$

The equations just preceding enable us to replace this inequality by

$$\frac{k^2\nu_3^2}{m_1m_2} + \frac{k^2\lambda_3^2}{m_2m_3} + \frac{k^2\mu_3^2}{m_3m_1} > 0;$$

and from this we conclude that

$$\frac{m_1\lambda_3^2 + m_2\mu_3^2 + m_3\nu_3^2}{m_1m_2m_3} > 0,$$

and finally on account of (3_3), that $m_1m_2m_3 > 0$. Hence, either $m_1 > 0$, $m_2 > 0$ and $m_3 > 0$, in which case our objective has been reached, or else one of these numbers is positive, the other two negative. Let us suppose $m_1 > 0$ and $m_2 < 0$, $m_3 < 0$. We derive then from (3_1) and (3_2), the inequalities

$$m_1\lambda_1^2 > -m_2\mu_1^2 - m_3\nu_1^2 > 0 \quad \text{and} \quad m_1\lambda_2^2 > -m_2\mu_2^2 - m_3\nu_2^2 > 0;$$

multiplication of these inequalities leads to

$$m_1^2\lambda_1^2\lambda_2^2 > m_2^2\mu_1^2\mu_2^2 + m_2m_3(\mu_1^2\nu_2^2 + \mu_2^2\nu_1^2) + m_3^2\nu_1^2\nu_2^2.$$

If from the two sides of this inequality we subtract the equation

$$m_1^2\lambda_1^2\lambda_2^2 = m_2^2\mu_1^2\mu_2^2 + 2\, m_2m_3\mu_1\mu_2\nu_1\nu_2 + m_3^2\nu_1^2\nu_2^2$$

obtained from (4_3) we reach the inequality

$$0 > m_2m_3(\mu_1\nu_2 - \mu_2\nu_1)^2,$$

from which would follow that $m_2m_3 < 0$ and therefore that m_2 and m_3 are opposite in sign; but this contradicts the supposition which we started from. We conclude therefore that $m_1 > 0$, $m_2 > 0$ and $m_3 > 0$; and we have the following theorem.

THEOREM 4. **The ellipsoid is the only central quadric for which there exist sets of conjugate diameters each of which meets the surface in real finite points.**

In developing further properties of the conjugate sets of diameters of the ellipsoid, we shall use equation (1), Section 115, with the understanding that $m_1 = \dfrac{1}{p^2}$, $m_2 = \dfrac{1}{q^2}$ and $m_3 = \dfrac{1}{r^2}$.

If d_1, d_2, and d_3 are conjugate diameters of the ellipsoid, we derive from the equations

$$(m_1\lambda_i^2 + m_2\mu_i^2 + m_3\nu_i^2)\delta_i^2 = 1 \quad \text{and} \quad m_1\lambda_i\lambda_j + m_2\mu_i\mu_j + m_3\nu_i\nu_j = 0, \quad i, j = 1, 2, 3, i \neq j,$$

the interpretation that

$$\begin{array}{lll} \lambda_1\delta_1\sqrt{m_1}, & \mu_1\delta_1\sqrt{m_2}, & \nu_1\delta_1\sqrt{m_3}, \\ \lambda_2\delta_2\sqrt{m_1}, & \mu_2\delta_2\sqrt{m_2}, & \nu_2\delta_2\sqrt{m_3}, \end{array}$$

and

$$\lambda_3\delta_3\sqrt{m_1}, \quad \mu_3\delta_3\sqrt{m_2}, \quad \nu_3\delta_3\sqrt{m_3}$$

are the direction cosines of three mutually perpendicular lines. If we apply to them the results obtained in Theorem 6, Chapter V, the Remark 3 following this theorem, and Theorem 7, Chapter V (see Section 65, pages 123 and 124), we find the following further relations:

$$(5)\quad \delta_1{}^2\lambda_1\mu_1 + \delta_2{}^2\lambda_2\mu_2 + \delta_3{}^2\lambda_3\mu_3 = 0, \quad \delta_1{}^2\mu_1\nu_1 + \delta_2{}^2\mu_2\nu_2 + \delta_3{}^2\mu_3\nu_3 = 0, \quad \delta_1{}^2\nu_1\lambda_1 + \delta_2{}^2\nu_2\lambda_2 + \delta_3{}^2\nu_3\lambda_3 = 0;$$

$$(6)\quad m_1(\delta_1{}^2\lambda_1{}^2+\delta_2{}^2\lambda_2{}^2+\delta_3{}^2\lambda_3{}^2)=1, \quad m_2(\delta_1{}^2\mu_1{}^2+\delta_2{}^2\mu_2{}^2+\delta_3{}^2\mu_3{}^2)=1, \quad m_3(\delta_1{}^2\nu_1{}^2 + \delta_2{}^2\nu_2{}^2 + \delta_3{}^2\nu_3{}^2) = 1;$$

$$(7)\qquad \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix} \cdot \delta_1\delta_2\delta_3\sqrt{m_1m_2m_3} = \pm 1.$$

From these formulas we derive the following interesting results.

THEOREM 5. **The sum of the squares of the semi-diameters of a set of conjugate diameters of an ellipsoid is the same for all conjugate sets of diameters.**

Proof. The semi-diameters of the conjugate set are δ_1, δ_2, and δ_3. If we divide the three equations in (6) by m_1, m_2, and m_3 respectively and add the results, we find:

$$\delta_1{}^2(\lambda_1{}^2 + \mu_1{}^2 + \nu_1{}^2) + \delta_2{}^2(\lambda_2{}^2 + \mu_2{}^2 + \nu_2{}^2) + \delta_3{}^2(\lambda_3{}^2 + \mu_3{}^2 + \nu_3{}^2) = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3};$$

that is

$$\delta_1{}^2 + \delta_2{}^2 + \delta_3{}^2 = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = p^2 + q^2 + r^2.$$

THEOREM 6. **The volume of the parallelopiped of which the three semi-diameters of a conjugate set are coterminous edges is the same for all conjugate sets.**

Proof. To determine the required volume we have at our disposal the formula in Corollary 2 of Theorem 3, Chapter V (Section

62, page 118). In the present case, the symbols used in this formula have the following values:

$$a = \delta_1,\quad b = \delta_2,\quad c = \delta_3,\quad \cos\gamma = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2,\quad \cos\beta = \lambda_3\lambda_1 + \mu_3\mu_1 + \nu_3\nu_1,\quad \cos\alpha = \lambda_2\lambda_3 + \mu_2\mu_3 + \nu_2\nu_3.$$

If we make use now of formula (7) above and observe that

$$\begin{vmatrix} 1 & \lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2 & \lambda_1\lambda_3+\mu_1\mu_3+\nu_1\nu_3 \\ \lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2 & 1 & \lambda_2\lambda_3+\mu_2\mu_3+\nu_2\nu_3 \\ \lambda_1\lambda_3+\mu_1\mu_3+\nu_1\nu_3 & \lambda_2\lambda_3+\mu_2\mu_3+\nu_2\nu_3 & 1 \end{vmatrix} = \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}^2$$

(compare part (3) of the Remark following Theorem 16, Chapter I, Section 14, page 27), we obtain for the volume of the parallelopiped the expression

$$\pm\frac{\delta_1\delta_2\delta_3}{\delta_1\delta_2\delta_3\sqrt{m_1m_2m_3}} = \pm pqr.$$

THEOREM 7. **If the ellipsoid $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$ is referred to a reference frame O-$X'Y'Z'$, whose axes are the lines of a conjugate set of diameters, its equation is $\frac{x'^2}{\delta_1^2} + \frac{y'^2}{\delta_2^2} + \frac{z'^2}{\delta_3^2} = 1$, where δ_1, δ_2, and δ_3 are the semi-diameters of the conjugate set.**

Proof. Let $P(\alpha, \beta, \gamma)$ be an arbitrary point on the ellipsoid, as referred to the frame O-XYZ, so that $m_1\alpha^2 + m_2\beta^2 + m_3\gamma^2 = 1$; the coördinates α', β', γ' of P with respect to O-$X'Y'Z'$ are the lengths of the segments $P_{y'z'}P$, $P_{z'x'}P$, and $P_{x'y'}P$ of the lines through P parallel to OX', OY', and OZ' respectively (compare Fig. 14, page 113). The equations of the line through P parallel to OX' are

$$x = \alpha + \lambda_1 s,\quad y = \beta + \mu_1 s,\quad z = \gamma + \nu_1 s.$$

The equation of the plane $Y'OZ'$ is $m_1\lambda_1 x + m_2\mu_1 y + m_3\nu_1 z = 0$; and the distance $PP_{y'z'}$ is the value of s determined by the equation which results when the expressions for x, y, and z just preceding are substituted in this equation, that is, by the equation

$$m_1\lambda_1\alpha + m_2\mu_1\beta + m_3\nu_1\gamma + (m_1\lambda_1^2 + m_2\mu_1^2 + m_3\nu_1^2)s = 0.$$

Therefore

$$\alpha' = P_{y'z'}P = -PP_{y'z'} = \frac{m_1\lambda_1\alpha + m_2\mu_1\beta + m_3\nu_1\gamma}{m_1\lambda_1^2 + m_2\mu_1^2 + m_3\nu_1^2}$$
$$= \delta_1^2(m_1\lambda_1\alpha + m_2\mu_1\beta + m_3\nu_1\gamma).$$

In similar manner we find

$$\beta' = \delta_2^2(m_1\lambda_2\alpha + m_2\mu_2\beta + m_3\nu_2\gamma), \quad \gamma' = \delta_3^2(m_1\lambda_3\alpha + m_2\mu_3\beta + m_3\nu_3\gamma).$$

Consequently we find

$$\frac{\alpha'^2}{\delta_1^2} + \frac{\beta'^2}{\delta_2^2} + \frac{\gamma'^2}{\delta_3^2} = \delta_1^2(m_1\lambda_1\alpha + m_2\mu_1\beta + m_3\nu_1\gamma)^2$$
$$+ \delta_2^2(m_1\lambda_2\alpha + m_2\mu_2\beta + m_3\nu_2\gamma)^2 + \delta_3^2(m_1\lambda_3\alpha + m_2\mu_3\beta + m_3\nu_3\gamma)^2.$$

If the squares of the trinomials on the right side of this equation are expanded, we obtain a homogeneous polynomial of the second degree in α, β, and γ. The coefficient of α^2 is found to be $m_1^2(\lambda_1^2\delta_1^2 + \lambda_2^2\delta_2^2 + \lambda_3^2\delta_3^2) = m_1$, by virtue of formula ($6_1$); in similar manner we find that the coefficients of β^2 and γ^2 are m_2 and m_3 respectively. For the coefficient of $\beta\gamma$ we find $2\, m_2 m_3 (\mu_1\nu_1\delta_1^2 + \mu_2\nu_2\delta_2^2 + \mu_3\nu_3\delta_3^2) = 0$, on account of formula (5_2); and it should be a simple matter to verify that the coefficients of $\gamma\alpha$ and $\alpha\beta$ are likewise zero. Hence we find

$$\frac{\alpha'^2}{\delta_1^2} + \frac{\beta'^2}{\delta_2^2} + \frac{\gamma'^2}{\delta_3^2} = m_1\alpha^2 + m_2\beta^2 + m_3\gamma^2 = 1.$$

Since α', β', and γ' are the coördinates with respect to $O\text{–}X'Y'Z'$ of an arbitrary point on the ellipsoid, our theorem is proved.

118. Exercises.

1. Prove that a set of conjugate diameters of the hyperboloid of one sheet $m_1x^2 + m_2y^2 + m_3z^2 = 1$ ($m_1 > 0$, $m_2 > 0$, $m_3 < 0$) is also a conjugate set for the hyperboloid of two sheets $m_1x^2 + m_2y^2 + m_3z^2 = -1$.

2. Prove that of a set of conjugate diameters of the two hyperboloids of the preceding exercise, two and only two meet one of these surfaces in real, finite points, whereas the third diameter of the set meets the other surface in real, finite points; and that this set of conjugate diameters for the two surfaces determines three finite chords, two chords of one of the surfaces, and one of the other.

3. Prove that if $2\,\delta_1$, $2\,\delta_2$, and $2\,\delta_3$ denote the lengths of the diameters of a conjugate set for the two hyperboloids of Exercise 1, then $\delta_1^2 + \delta_2^2 + \delta_3^2 = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}$.

4. Determine the diameter of the surfaces $x^2 - \frac{y^2}{4} + \frac{z^2}{9} = \pm 1$, which is conjugate to the two diameters whose direction cosines are proportional to $-1 : 6 : 9$ and to $1 : 2 : 4$ respectively.

5. Prove that three tangent planes of an ellipsoid which are parallel to the three diametral planes of a conjugate set meet in a point. Determine the locus generated by this point when all conjugate sets of diametral planes are considered.

6. Determine the length of the diameter conjugate to two diameters which lie in a plane of circular section through the center of an ellipsoid.

119. Linear Families of Quadrics. When two quadric surfaces are given, as for instance by the equations $Q(x, y, z) = 0$ and $Q'(x, y, z) = 0$, the points common to these surfaces determine a space locus with one degree of freedom, that is, a space curve. The study of space curves constitutes an important part of the field of Differential Geometry. Without knowing anything further about the character of the curve of intersection of two quadrics, we can affirm that, no matter what polynomials are represented by the symbols $A(x, y, z)$ and $B(x, y, z)$, the locus of the equation

$$A(x, y, z) \cdot Q(x, y, z) + B(x, y, z) \cdot Q'(x, y, z) = 0$$

will be a surface which passes through this curve; for this equation is surely satisfied by the coördinates of any point which belongs to the locus of $Q = 0$ *and* to that of $Q' = 0$. And if $A(x, y, z)$ and $B(x, y, z)$ reduce to constants which are not both zero, the equation represents a quadric through this curve.

Thus the equation

$$(1)\quad k_1 Q(x, y, z) + k_2 Q'(x, y, z) = 0$$

represents, when the ratio $k_1 : k_2$ varies, a family of quadric surfaces, all of which pass through the points common to the surfaces $Q = 0$ and $Q' = 0$. It is called a linear one-parameter family of quadrics, also a pencil of quadrics (compare Section 49). For $k_1 = 0$, $k_2 = 1$, we obtain the surface Q'; for $k_1 = 1$, $k_2 = 0$, the surface Q. Upon division by k_1 and putting $\frac{k_2}{k_1} = -\lambda$, the equation (1) takes the form

$$(2)\quad Q(x, y, z) - \lambda Q'(x, y, z) = 0;$$

and this equation is equivalent to (1), except that it does not include the surface Q' for any finite value of λ.

The value of the parameter λ may be so selected as to make the surface represented by (2) satisfy a single condition, for example, that of passing through one prescribed point, which does not lie on the surface Q'. This particular problem has a unique solution. For if (α, β, γ) is the prescribed point, λ must satisfy the condition

$$Q(\alpha, \beta, \gamma) - \lambda Q'(\alpha, \beta, \gamma) = 0,$$

so that

$$\lambda = \frac{Q(\alpha, \beta, \gamma)}{Q'(\alpha, \beta, \gamma)}.$$

The condition that a surface of the pencil be a surface of revolution leads to an equation of higher degree in λ; for it requires that λ be determined so that the equation

$$\begin{vmatrix} a_{11} - \lambda a_{11}' - k & a_{12} - \lambda a_{12}' & a_{13} - \lambda a_{13}' \\ a_{12} - \lambda a_{12}' & a_{22} - \lambda a_{22}' - k & a_{23} - \lambda a_{23}' \\ a_{13} - \lambda a_{13}' & a_{23} - \lambda a_{23}' & a_{33} - \lambda a_{33}' - k \end{vmatrix} = 0$$

shall have a double root (compare the Remark on page 215). We shall not pursue this problem further.

Of special interest is the question as to the singular quadrics contained in the pencil of quadrics (2). A surface of this pencil will be singular if and only if the rank of the matrix

$$\mathbf{a}_4(\lambda) = \begin{Vmatrix} a_{11} - \lambda a_{11}' & a_{12} - \lambda a_{12}' & a_{13} - \lambda a_{13}' & a_{14} - \lambda a_{14}' \\ a_{12} - \lambda a_{12}' & a_{22} - \lambda a_{22}' & a_{23} - \lambda a_{23}' & a_{24} - \lambda a_{24}' \\ a_{13} - \lambda a_{13}' & a_{23} - \lambda a_{23}' & a_{33} - \lambda a_{33}' & a_{34} - \lambda a_{34}' \\ a_{14} - \lambda a_{14}' & a_{24} - \lambda a_{24}' & a_{34} - \lambda a_{34}' & a_{44} - \lambda a_{44}' \end{Vmatrix}$$

is less than 4 (compare Definition V, Chapter VII, Section 82, page 166); it will be a non-degenerate singular quadric, that is, a proper cone or a cylinder, if and only if the rank of this matrix is 3. In either case a necessary condition is that the determinant $\Delta(\lambda)$ of this matrix shall vanish. But the equation $\Delta(\lambda) = 0$ is of the fourth degree in λ; there will therefore be at most four singular quadrics in the pencil. A further discussion of the characteristics of these surfaces, of the questions whether they are degenerate surfaces, whether they are cylinders or cones leads into a more extensive treatment than we can undertake here. The interested reader is referred to Snyder and Sisam, Analytic Geom-

etry of Space, Chapter XI, or to Bôcher, Introduction to Higher Algebra, Chapter XIII, for a consideration of these problems.

To determine the ratios of the ten coefficients which appear in the general equation $Q(x, y, z) = 0$ of a quadric surface, nine points must be prescribed. Substitution of the coördinates of these nine points in the general equation furnishes nine linear homogeneous equations for a_{11}, a_{22}, a_{33}, a_{44}, a_{12}, a_{23}, a_{34}, a_{14}, a_{13}, a_{24} from which the ratios of these coefficients can in general be determined. In order that this be possible, the 9 points $(\alpha_i, \beta_i, \gamma_i)$, $i = 1, 2, \ldots, 9$ must have such coördinates that the rank of the matrix, formed by the 9 rows

$$(3) \quad \alpha_i^2, \quad \beta_i^2, \quad \gamma_i^2, \quad 2\,\beta_i\gamma_i, \quad 2\,\gamma_i\alpha_i, \quad 2\,\alpha_i\beta_i, \quad 2\,\alpha_i, \quad 2\,\beta_i, \quad 2\,\gamma_i, \quad 1$$

$(i = 1, 2, \ldots, 9)$ of 10 elements each, is 9. Whenever a set of points $(\alpha_i, \beta_i, \gamma_i)$ is so constituted that the matrix, in which for each of the points there is a row of 10 elements as indicated in (3), has a rank equal to the number of points in the set, we shall say that these points are in "general position." We can therefore say that there passes a single quadric surface through a set of nine points in "general position."

If only eight points are given, we have eight equations; if the eight points are in general position, these equations will enable us to determine 8 of the coefficients a_{ij} as linear homogeneous functions of the other two, by Cramer's rule; if these two be called k_1 and k_2, the solution will be of the form $a_{ij} = a_{ij}'k_1 + a_{ij}''k_2$, where a_{ij}' and a_{ij}'' are known. The equation of the quadric through the given 8 points will then be $Q(x, y, z) = 0$, where $Q(x, y, z) = k_1Q'(x, y, z) + k_2Q''(x, y, z)$. The surface $Q = 0$ will therefore belong to a pencil of quadrics determined by the quadrics $Q' = 0$ and $Q'' = 0$, provided the latter two surfaces are distinct. It will be worth our while to inquire further whether this will indeed be the case. If we suppose that the value of the determinant of 8th order, which is formed by the first 8 elements in (3) for $i = 1, 2, \ldots, 8$, is different from zero (let us denote this determinant by D_8), then we may take $k_1 = a_{34}$, $k_2 = a_{44}$; and it follows from Cramer's rule that a_{11} is equal to a fraction whose denominator is equal to D_8; and the numerator is obtained from the denominator by writing the column $2\,a_{34}\gamma_i + a_{44} (i = 1, 2, \ldots, 8)$ in place of the column α_i^2, $i = 1, 2, \ldots, 8$. Hence if we denote by D_8'

and D_8'' the determinants obtained from D_8 by replacing its first column by $2\gamma_i$, $i = 1, 2, \ldots, 8$ and by a column of ones respectively, we find that $a_{11} = \frac{D_8'}{D_8} \times a_{34} + \frac{D_8''}{D_8} \times a_{44} = k_1 a_{11}' + k_2 a_{11}''$, where $a_{11}' = \frac{D_8'}{D_8}$ and $a_{11}'' = \frac{D_8''}{D_8}$. Similarly we find $a_{22} = k_1 a_{22}' + k_2 a_{22}''$, where a_{22}' and a_{22}'' are obtained from D_8 by replacing the second column by the columns which were used in the first column for the formation of D_8' and D_8'' respectively, and dividing by D_8; and so on for all the coefficients a_{ij} except a_{34} and a_{44}. Finally, if we write $a_{34}' = 1$, $a_{34}'' = 0$, and $a_{44}' = 0$, $a_{44}'' = 1$, we have also $a_{34} = k_1 a_{34}' + k_2 a_{34}''$, and $a_{44} = k_1 a_{44}' + k_2 a_{44}''$. Hence we can write $Q(x, y, z) = k_1 Q'(x, y, z) + k_2 Q''(x, y, z)$, where the coefficients of Q' are not proportional to those of Q''. It should be clear that the same conclusion will be reached if we start from the supposition that another 8th order determinant of the matrix is different from zero. Therefore any quadric Q which passes through the 8 given points belongs to the pencil of quadrics determined by the quadrics Q' and Q'' and passes through the curve of intersection of these surfaces.

THEOREM 8. **All quadrics which have in common eight points in general position have a curve in common which passes through these eight points.**

A system of three quadrics $Q(x, y, z) = 0$, $Q'(x, y, z) = 0$ and $Q''(x, y, z) = 0$ which do not belong to one pencil, that is, such that the only values of k_1, k_2, and k, for which the relation $kQ(x, y, z) \equiv k_1 Q'(x, y, z) + k_2 Q''(x, y, z)$ holds identically are $k_1 = k_2 = k = 0$, determines a linear two-parameter family of quadrics $kQ + k_1 Q' + k_2 Q'' = 0$, or $Q + \lambda Q' + \mu Q'' = 0$. Such a family is called a bundle of quadrics. All surfaces of the bundle pass through the points common to the three initial surfaces. These equations will in general have 8 points in common, since their equations are of the second degree in x, y, z. But these 8 points are not in general position, since otherwise the three surfaces Q, Q' and Q'' would, in accordance with Theorem 8, belong to one pencil.

Suppose now that there be given 7 points in general position; then seven of the coefficients a_{ij} can be determined in terms of the

remaining three; if these be called k_1, k_2, and k_3, we shall find $a_{ij} = k_1a_{ij}' + k_2a_{ij}'' + k_3a_{ij}'''$, and therefore $Q(x, y, z) = k_1Q'(x, y, z) + k_2Q''(x, y, z) + k_3Q'''(x, y, z)$. The determination of the functions Q', Q'', and Q''' proceeds by a method analogous to the one used in the proof of Theorem 8. On the supposition that the determinant of 7th order whose rows are formed by the first seven elements of (3) for $i = 1, 2, \ldots, 7$ does not vanish, we may take $k_1 = a_{24}$, $k_2 = a_{34}$, $k_3 = a_{44}$. Therefore if $a_{ij} = k_1a_{ij}' + k_2a_{ij}'' + k_3a_{ij}'''$, we have $a_{24}' = 1$, $a_{24}'' = 0$, $a_{24}''' = 0$; $a_{34}' = 0$, $a_{34}'' = 1$, $a_{34}''' = 0$; $a_{44}' = 0$, $a_{44}'' = 0$, $a_{44}''' = 1$. This shows that the three surfaces $Q' = 0$, $Q'' = 0$, and $Q''' = 0$ do not belong to the same pencil. Therefore any surface through the 7 given points belongs to the bundle of quadrics determined by these three surfaces and will therefore pass through the eight points common to them.

THEOREM 9. **All quadrics which have in common 7 points in general position also have an eighth point in common.**

120. Focal Curves and Directrix Cylinders of Central Quadrics.

In Plane Analytical Geometry, the focus and directrix of a conic section are defined as a point and a line such that the ratio of the distances of any point on the curve from the focus and from the directrix is the same as for any other point on the curve. A corresponding definition of focus and directrix for the quadric surfaces seems to have been given first by Chasles (in 1835) or perhaps by Salmon (in 1862).

DEFINITION IV. **A focus and a directrix of a quadric surface are a point and a line such that the ratio of the square of the distance of any point P on the surface from the focus to the product of its distances from any two planes through the directrix is constant.***

Although the concepts defined in this manner are entirely definite, it is still an open question whether quadric surfaces possess foci and directrices. We shall not deal with this question for the general quadric but only for the central quadrics, which are not surfaces of revolution.

* The planes through the directrix need not be real. The line of intersection of two planes may be real even though the planes themselves are not real; for example, the planes $x + iy - z = 0$ and $x - iy - z = 0$ both contain the line $x = t$, $y = 0$, $z = t$.

If $F(\alpha, \beta, \gamma)$ is a focus of the central quadric

$$(1)\quad m_1x^2 + m_2y^2 + m_3z^2 = 1$$

(m_1, m_2, and m_3 being distinct and all different from zero), there must exist two linear functions $l(x, y, z)$ and $l'(x, y, z)$, and a constant c such that

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = c \times l(x, y, z) \times l'(x, y, z)$$

for every set of values of x, y, z which satisfy equation (1).

Consequently the functions $m_1x^2 + m_2y^2 + m_3z^2 - 1$ and $(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 - cll'$ must differ by a constant factor λ; that is, there must exist a factor λ such that

$$m_1x^2 + m_2y^2 + m_3z^2 - 1 - \lambda[(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2] \\ = \lambda c l(x, y, z) l'(x, y, z).$$

Therefore the function on the left-hand side of this identity must be equal to the product of two factors which are linear in x, y, and z and the directrix which corresponds to the focus $F(\alpha, \beta, \gamma)$ will then be the line of intersection of the planes represented by the equations which result when these linear factors are equated to zero. According to Theorem 6, Chapter VIII (Section 96, page 206) the necessary and sufficient condition for the possibility of thus factoring the function is that the rank of the discriminant matrix of the quadric surface

$$m_1x^2 + m_2y^2 + m_3z^2 - 1 - \lambda[(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2] = 0$$

is less than 3. The necessary and sufficient conditions which will make certain that the point $F(\alpha, \beta, \gamma)$ is a focus of the quadric Q are therefore that the coördinates α, β, γ cause the determinant

$$\Delta(\lambda, \alpha, \beta, \gamma) = \begin{vmatrix} m_1 - \lambda & 0 & 0 & \lambda\alpha \\ 0 & m_2 - \lambda & 0 & \lambda\beta \\ 0 & 0 & m_3 - \lambda & \lambda\gamma \\ \lambda\alpha & \lambda\beta & \lambda\gamma & -[1 + \lambda(\alpha^2 + \beta^2 + \gamma^2)] \end{vmatrix}$$

and all its three-rowed minors to vanish. But because the matrix of this determinant is symmetric, the second condition may be replaced, in view of Theorem 6, Chapter II (Section 26, page 43), by the requirement that the principal three-rowed minors vanish.

If the principal three-rowed minor in the upper left-hand corner

is to vanish, we must have $(m_1 - \lambda)(m_2 - \lambda)(m_3 - \lambda) = 0$, that is, $\lambda = m_1$, or $\lambda = m_2$, or $\lambda = m_3$.

Let us consider the case $\lambda = m_1$. The principal minor formed from the 1st, 2nd, and 4th columns and rows then becomes

$$\begin{vmatrix} 0 & 0 & m_1\alpha \\ 0 & m_2 - m_1 & m_1\beta \\ m_1\alpha & m_1\beta & -[1 + m_1(\alpha^2 + \beta^2 + \gamma^2)] \end{vmatrix} = -m_1{}^2\alpha^2(m_2 - m_1).$$

Since $m_1 \neq 0$ and $m_2 \neq m_1$, the vanishing of this determinant requires that $\alpha = 0$; and when $\lambda = m_1$ and $\alpha = 0$, the fourth-order determinant and the three-rowed principal minor formed from the 1st, 3rd, and 4th columns and rows obviously vanish as well. There remains therefore the condition that the three-rowed principal minor in the lower right-hand corner shall vanish, that is, that

$$\begin{vmatrix} m_2 - m_1 & 0 & m_1\beta \\ 0 & m_3 - m_1 & m_1\gamma \\ m_1\beta & m_1\gamma & -[1 + m_1(\beta^2 + \gamma^2)] \end{vmatrix} = 0.$$

If we develop this determinant, we are led to the equation $m_1m_2(m_3 - m_1)\beta^2 + m_1m_3(m_2 - m_1)\gamma^2 + (m_2 - m_1)(m_3 - m_1) = 0$; we write this equation in the form

$$(2) \qquad \frac{\beta^2}{\dfrac{1}{m_1} - \dfrac{1}{m_2}} + \frac{\gamma^2}{\dfrac{1}{m_1} - \dfrac{1}{m_3}} + 1 = 0.$$

This equation, in conjunction with the equation $\alpha = 0$, determines for varying β and γ a real or imaginary conic section in the YZ-plane; it is called a **focal conic** of the quadric surface.

If, conversely, (α, β, γ) is a point on this conic section,

$$\begin{aligned} m_1x^2 + m_2y^2 + m_3z^2 - 1 - m_1[x^2 + (y - \beta)^2 + (z - \gamma)^2] \\ = (m_2 - m_1)y^2 + (m_3 - m_1)z^2 + 2\,m_1\beta y + 2\,m_1\gamma z - m_1\beta^2 \\ -m_1\gamma^2 - 1 \\ = \frac{1}{m_2 - m_1} \cdot [(m_2 - m_1)y + m_1\beta]^2 \\ + \frac{1}{m_3 - m_1} \cdot [(m_3 - m_1)z + m_1\gamma]^2; \end{aligned}$$

and this function is factorable into two factors linear in y and z (the variable x is absent), with coefficients which are real or imagi-

nary according as $m_2 - m_1$ and $m_3 - m_1$ are of opposite signs or of like signs. In either case, the equations which are obtained by equating these factors to zero represent planes which pass through the line determined by the equations

$$(3)\quad (m_2 - m_1)y + m_1\beta = 0 \quad \text{and} \quad (m_3 - m_1)z + m_1\gamma = 0.$$

This line and the point (α, β, γ) on the conic section determined above are therefore a directrix and a focus of the quadric surface (1). As the point (α, β, γ) describes the conic (2), the line (3), which is parallel to the X-axis describes a cylindrical surface, whose equation is obtained when β and γ are eliminated from equations (2) and (3). This cylindrical surface is called a **directrix cylinder** of the quadric. We conclude therefore that from the value $\lambda = m_1$, we obtain the focal conic

$$(4)\quad \frac{y^2}{\frac{1}{m_1} - \frac{1}{m_2}} + \frac{z^2}{\frac{1}{m_1} - \frac{1}{m_3}} + 1 = 0, \quad x = 0,$$

and the directrix cylinder

$$(5)\quad m_2(m_1 - m_2)\frac{y^2}{m_1} + m_3(m_1 - m_3)\frac{z^2}{m_1} = 1.$$

The reader should have no difficulty in showing that from $\lambda = m_2$ and $\lambda = m_3$, we obtain the focal conics represented respectively by the pairs of equations

$$(6)\quad \frac{z^2}{\frac{1}{m_2} - \frac{1}{m_3}} + \frac{x^2}{\frac{1}{m_2} - \frac{1}{m_1}} + 1 = 0, \quad y = 0,$$

and

$$(7)\quad \frac{x^2}{\frac{1}{m_3} - \frac{1}{m_1}} + \frac{y^2}{\frac{1}{m_3} - \frac{1}{m_2}} + 1 = 0, \quad z = 0;$$

and the corresponding directrix cylinders given respectively by the equations

$$(8)\quad m_3(m_2 - m_3)\frac{z^2}{m_2} + m_1(m_2 - m_1)\frac{x^2}{m_2} = 1,$$

and

$$(9) \quad m_1(m_3 - m_1)\frac{x^2}{m_3} + m_2(m_3 - m_2)\frac{y^2}{m_3} = 1.$$

The results of this discussion are summarized in the theorem which follows.

THEOREM 10. **For every central quadric surface which is not a surface of revolution there exist three real or imaginary focal conics and three corresponding directrix cylinders; with every point on a focal conic there is associated a generating line of a directrix cylinder, so that point and line are focus and directrix of the quadric, as defined in Definition IV.**

121. Focal Conics and Directrix Cylinders, *continued.* We turn now to a further consideration of the focal conics and directrix cylinders for each type of central quadric, particularly with a view to determining the conditions under which they are real.

CASE I. *Ellipsoid.*

If we take the equation in the standard form

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1, \quad p < q < r,$$

we have $m_1 = \dfrac{1}{p^2}$, $m_2 = \dfrac{1}{q^2}$, and $m_3 = \dfrac{1}{r^2}$; and $\dfrac{1}{m_1} - \dfrac{1}{m_2} = p^2 - q^2 < 0$, $\dfrac{1}{m_1} - \dfrac{1}{m_3} = p^2 - r^2 < 0$. Hence equations (4) and (5) of the preceding section give the real focal ellipse

$$x = 0, \ \frac{y^2}{q^2 - p^2} + \frac{z^2}{r^2 - p^2} = 1,$$

and the corresponding real elliptic directrix cylinder

$$(q^2 - p^2)\frac{y^2}{q^4} + (r^2 - p^2)\frac{z^2}{r^4} = 1.$$

Equations (6) and (8) lead to the real focal hyperbola

$$y = 0, \quad \frac{z^2}{r^2 - q^2} - \frac{x^2}{q^2 - p^2} = 1,$$

and the associated real hyperbolic directrix cylinder

$$(r^2 - q^2)\frac{z^2}{r^4} - (q^2 - p^2)\frac{x^2}{p^4} = 1.$$

Finally, equations (7) and (9) lead to an imaginary focal ellipse and an imaginary elliptic directrix cylinder. We reach therefore the following conclusion:

THEOREM 11. **An ellipsoid possesses a real focal ellipse in the principal plane determined by the two longer semi-axes, and a real focal hyperbola in the principal plane determined by the two extreme semi-axes. The associated directrix cylinders are real, elliptic and hyperbolic respectively, their generators being perpendicular to the planes of the corresponding focal curves.**

CASE II. *Hyperboloid of One Sheet.*

With the standard equation in the form

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1, \quad p < q,$$

we find

$$m_1 = \frac{1}{p^2}, \quad m_2 = \frac{1}{q^2}, \quad m_3 = -\frac{1}{r^2}.$$

From equations (4) and (5) of Section 120, we obtain the real focal hyperbola

$$x = 0, \quad \frac{y^2}{q^2 - p^2} - \frac{z^2}{p^2 + r^2} = 1,$$

and the real hyperbolic directrix cylinder

$$(q^2 - p^2)\frac{y^2}{q^4} - (p^2 + r^2)\frac{z^2}{r^4} = 1.$$

Equations (7) and (9) give the real focal ellipse

$$z = 0, \quad \frac{x^2}{p^2 + r^2} + \frac{y^2}{q^2 + r^2} = 1,$$

and the real elliptic directrix cylinder

$$(p^2 + r^2)\frac{x^2}{p^4} + (q^2 + r^2)\frac{y^2}{q^4} = 1.$$

The loci determined by equations (6) and (8) are imaginary in this case. The following theorem states the results.

THEOREM 12. **An hyperboloid of one sheet possesses a real focal ellipse in the principal plane which cuts the surface in an ellipse, and a real focal hyperbola in the principal plane determined by the conjugate axis and the longer of the two transverse semi-axes. The associated directrix cylinders are real, elliptic and hyperbolic respectively, with generators perpendicular to the planes of the corresponding focal curves.**

CASE III. *Hyperboloid of Two Sheets.*

The standard form of the equation

$$\frac{x^2}{p^2} - \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1, \quad q < r$$

gives

$$m_1 = \frac{1}{p^2}, \quad m_2 = -\frac{1}{q^2}, \quad m_3 = -\frac{1}{r^2}.$$

Equations (6) and (8) yield the real focal hyperbola

$$y = 0, \quad \frac{x^2}{p^2 + q^2} - \frac{z^2}{r^2 - q^2} = 1,$$

and the real hyperbolic directrix cylinder

$$(p^2 + q^2)\frac{x^2}{p^4} - (r^2 - q^2)\frac{z^2}{r^4} = 1.$$

From equations (7) and (9) we obtain the real focal ellipse

$$z = 0, \quad \frac{x^2}{p^2 + r^2} + \frac{y^2}{r^2 - q^2} = 1,$$

and the real elliptic directrix cylinder

$$(p^2 + r^2)\frac{x^2}{p^4} + (r^2 - q^2)\frac{y^2}{q^4} = 1.$$

The loci determined by equations (4) and (5) are imaginary in this case. The conclusions are therefore as stated in the next theorem.

THEOREM 13. **An hyperboloid of two sheets possesses a real focal ellipse in the principal plane determined by the transverse axis and the shorter of the two conjugate semi-axes, and a real focal hyperbola in the principal plane determined by the transverse axis and the longer conjugate semi-axis; the associated directrix cylinders are real, elliptic and hyperbolic respectively, with generators perpendicular to the planes of the corresponding focal curves.**

122. Exercises.

1. Determine the focal curves and the directrix cylinders of each of the following surfaces:

(*a*) $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{6} = 1,$ (*b*) $\frac{x^2}{9} + \frac{y^2}{6} - \frac{z^2}{4} = 1,$

(*c*) $\frac{x^2}{6} - \frac{y^2}{9} - \frac{z^2}{4} = 1.$

2. Prove that an ellipsoid of revolution has a real focal circle in the principal plane which cuts the surface in a circle and a real circular directrix cylinder.

3. Prove that an hyperboloid of revolution of one sheet has a real focal circle and a real circular directrix cylinder.

4. Prove that the focal ellipse of an ellipsoid is similar to the ellipse in which the surface is cut by the plane of the focal ellipse if and only if the surface is an ellipsoid of revolution.

5. Prove that the focal ellipse of an hyperboloid of one sheet is similar to the ellipse in which the surface is cut by the plane of the focal ellipse if and only if the hyperboloid is a surface of revolution.

6. Prove that the principal plane determined by the two shorter semi-axes of an ellipsoid cuts the focal ellipse and the associated directrix cylinder in a point and a line respectively which are the focus and the directrix of the ellipse in which the surface is cut by this plane.

7. Prove that the semi-axes of the ellipse in which an ellipsoid is cut by the plane of the focal ellipse are mean proportionals between the corresponding semi-axes of the focal ellipse and of the directrix curve of the associated elliptic directrix cylinder; also that the semi-axes of the ellipse in which an ellipsoid is cut by the plane of the focal hyperbola are mean proportionals between the corresponding semi-axes of the focal hyperbola and of the associated hyperbolic directrix cylinder.

8. Prove theorems analogous to those of the preceding exercise for the hyperboloid of one sheet and for the hyperboloid of two sheets.

9. Determine the distance from the origin of the points in which the focal hyperbola of an ellipsoid is met by the planes of central circular section.

10. Determine the conditions under which the focal hyperbola of a central quadric is a rectangular hyperbola.

11. Prove that the foci of the focal curves of an ellipsoid coincide with the foci of the conic sections in which the surface is cut by the planes of these focal curves.

12. Prove the corresponding theorem for the hyperboloids of one and two sheets.

123. Confocal Quadric Surfaces. Elliptic Coördinates. It follows from formulas (4), (6), and (7) of Section 120 (see page 284) that two central quadrics

$$m_1x^2 + m_2y^2 + m_3z^2 = 1 \quad \text{and} \quad m_1'x^2 + m_2'y^2 + m_3'z^2 = 1$$

will have their focal curves in common if and only if $\dfrac{1}{m_i} - \dfrac{1}{m_j} = \dfrac{1}{m_i'} - \dfrac{1}{m_j'}$, for $i, j = 1, 2, 3$.

This will certainly be the case therefore for all surfaces repre-

sented by the equation

$$(1)\quad \frac{x^2}{p^2-\lambda}+\frac{y^2}{q^2-\lambda}+\frac{z^2}{r^2-\lambda}=1, \quad p<q<r,$$

in which λ is a real parameter. For all these surfaces the focal ellipse is given by the equations

$$x=0, \quad \frac{y^2}{q^2-p^2}+\frac{z^2}{r^2-p^2}=1,$$

and the focal hyperbola by the equations

$$y=0, \quad \frac{z^2}{r^2-q^2}-\frac{x^2}{q^2-p^2}=1.$$

The family of surfaces represented by equation (1) is called a **confocal family of quadric surfaces.**

If $\lambda < p^2$, all the denominators in equation (1) are positive; the surface is therefore an ellipsoid. If $p^2 < \lambda < q^2$, the first denominator is negative, the other two are positive, so that the surface is an hyperboloid of one sheet, of which the X-axis is the conjugate axis. If $q^2 < \lambda < r^2$, the first two denominators are negative and the third one is positive; hence the surface is an hyperboloid of two sheets, of which the Z-axis is the transverse axis. Finally if $\lambda > r^2$, the surface is an imaginary ellipsoid.

For the critical values $\lambda = p^2$, $\lambda = q^2$, and $\lambda = r^2$, the equation (1) has no meaning. If we multiply both sides of equation (1) by $p^2 - \lambda$, we obtain the equation

$$(2)\quad x^2+(p^2-\lambda)\left[\frac{y^2}{q^2-\lambda}+\frac{z^2}{r^2-\lambda}-1\right]=0,$$

which is equivalent to (1) except when $\lambda = p^2$. For this value of λ, equation (2) reduces to $x^2 = 0$, whose locus is the YZ-plane counted doubly. If equation (1) is multiplied through by $q^2 - \lambda$ and by $r^2 - \lambda$, we obtain equations, which for $\lambda = q^2$ and $\lambda = r^2$ reduce respectively to the equations $y^2 = 0$ and $z^2 = 0$. We complete now the definition of the confocal family of quadrics given by equation (1) by the statement that to the values $\lambda = p^2$, $\lambda = q^2$, and $\lambda = r^2$ shall correspond the YZ-plane, the ZX-plane, and the XY-plane respectively, each counted doubly. The character of the surfaces in the confocal family is indicated diagrammatically in Figure 36.

We shall now try to determine in what manner the surfaces of the family change as λ tends towards the critical values, passing through values which remain steadily on one side of a critical value. To indicate that λ tends toward p^2 through values which are greater than p^2, we shall write $\lambda \to p^2 + 0$; to indicate that λ tends toward p^2 through values which are less than p^2, we shall write $\lambda \to p^2 - 0$. Similar meanings are to be attributed to the notations $\lambda \to q^2 + 0$, $\lambda \to q^2 - 0$, $\lambda \to r^2 + 0$ and $\lambda \to r^2 - 0$.

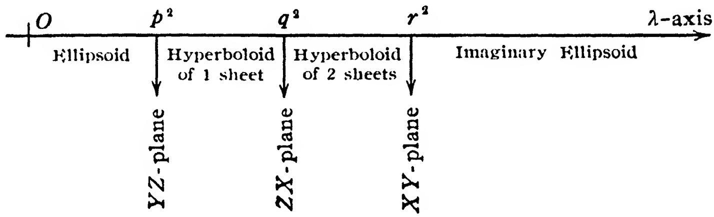

FIG. 36

As $\lambda \to p^2 - 0$, the surface is steadily an ellipsoid; its semi-axis along the X-axis tends to zero. Since $\lambda < p^2$, the factor $p^2 - \lambda$ in the second term of equation (2) is positive; it should therefore be clear from this equation that $\dfrac{y^2}{q^2 - \lambda} + \dfrac{z^2}{r^2 - \lambda} - 1 < 0$. But points in the YZ-plane for which this inequality holds lie on the inside of the ellipse $\dfrac{y^2}{q^2 - \lambda} + \dfrac{z^2}{r^2 - \lambda} = 1$.* Hence as $\lambda \to p^2 - 0$, the surface tends toward those points of the YZ-plane which lie on the inside of the focal ellipse

$$x = 0, \quad \frac{y^2}{q^2 - p^2} + \frac{z^2}{r^2 - p^2} = 1.$$

* To be convinced of this fact, it is only necessary to observe that at the origin, the function $\dfrac{y^2}{q^2 - \lambda} + \dfrac{z^2}{r^2 - \lambda} - 1$ reduces to -1 and that since the function is a continuous function of λ, y, and z for all values of λ which differ from q^2 and r^2, it cannot change from negative values to positive values without becoming zero. Since this can take place only *on* the ellipse, points for which the function is negative lie on the same side of the curve as the origin, that is, on the inside of the ellipse; and points for which it is positive lie on the outside of the ellipse.

As $\lambda \to p^2 + 0$, the surface is steadily an hyperboloid of one sheet, whose semi-axis along the X-axis tends toward zero. But now $p^2 - \lambda$ is negative, and therefore $\frac{y^2}{q^2 - \lambda} + \frac{z^2}{r^2 - \lambda} - 1 > 0.$ Therefore as $\lambda \to p^2 + 0$, the surface tends toward those points of the YZ-plane which lie outside the focal ellipse.

As $\lambda \to q^2 - 0$, the surface is always an hyperboloid of one sheet, whose semi-axis along the Y-axis tends to zero. For those values of λ, $q^2 - \lambda$ is positive; it follows then from an equation analogous to (2) that $\frac{x^2}{p^2 - \lambda} + \frac{z^2}{r^2 - \lambda} - 1$ is negative. An argument similar to the one made in the footnote on the preceding page shows that points for which this inequality holds lie on the same side of the hyperbola $\frac{x^2}{p^2 - \lambda} + \frac{z^2}{r^2 - \lambda} = 1$ as the origin. If we call this the inside of the hyperbola, we conclude that as $\lambda \to q^2 - 0$, the surface tends toward the points of the ZX-plane which lie on the inside of the focal hyperbola

$$y = 0, \quad \frac{z^2}{r^2 - q^2} - \frac{x^2}{q^2 - p^2} = 1.$$

And the same reasoning shows that as $\lambda \to q^2 + 0$, the surface tends toward the points of the ZX-plane which lie outside the focal hyperbola.

Finally, as $\lambda \to r^2 - 0$, the surface is an hyperboloid of two sheets, whose semi-axis along the Z-axis (that is, the transverse axis) tends to zero. Since $r^2 - \lambda > 0$, it follows that $\frac{x^2}{p^2 - \lambda} + \frac{y^2}{q^2 - \lambda} - 1 < 0$; but now $\lambda > p^2$ and $\lambda > q^2$ and therefore this inequality is satisfied by all points in the XY-plane. Consequently as $\lambda \to r^2 - 0$, the surface tends toward the entire XY-plane.

We summarize the discussion by a theorem.

THEOREM 14. **The equation $\frac{x^2}{p^2 - \lambda} + \frac{y^2}{q^2 - \lambda} + \frac{z^2}{r^2 - \lambda} = 1$, $p < q < r$, in which λ is a real parameter, represents a confocal family of quadric surfaces. As λ increases from negative infinity to p^2, the locus of the equation is an ellipsoid which tends toward the inside of the focal ellipse of the family; as λ increases from p^2 to q^2, the locus is an hyperboloid of one sheet tending from the outside of the focal ellipse to the inside of the focal hyperbola of the system; as λ increases**

from q^2 to r^2, the locus is an hyperboloid of two sheets, which tends from the outside of the focal hyperbola to the entire XY-plane. For $\lambda = p^2$, $\lambda = q^2$, and $\lambda = r^2$, the locus is respectively the YZ-plane, the ZX-plane, and the XY-plane, each counted doubly.

We shall now prove two properties of confocal families of quadrics.

THEOREM 15. **Through every point in space that does not lie on one of the coördinate planes, there pass three surfaces of every confocal family of quadrics, namely, an ellipsoid, an hyperboloid of one sheet and an hyperboloid of two sheets.**

Proof. Let $P(\alpha, \beta, \gamma)$ be an arbitrary point of space that does not lie on any coördinate plane; then α, β, and γ are all different from zero. If P is to lie on a surface of the confocal family represented by equation (1), the parameter λ must be so determined that

$$\frac{\alpha^2}{p^2 - \lambda} + \frac{\beta^2}{q^2 - \lambda} + \frac{\gamma^2}{r^2 - \lambda} - 1 = 0;$$

that is, λ must be a root of the equation

$$F(\lambda) = (\lambda - p^2)(\lambda - q^2)(\lambda - r^2) + (\lambda - q^2)(\lambda - r^2)\,\alpha^2 + (\lambda - r^2)$$
$$(\lambda - p^2)\beta^2 + (\lambda - p^2)\,(\lambda - q^2)\gamma^2 = 0.$$

This is a cubic equation in which the coefficient of λ^3 is $+1$; consequently for large positive values of λ, $F(\lambda) > 0$ and for large negative values of λ, $F(\lambda) < 0$. Moreover

$$F(p^2) = (p^2 - q^2)\,(p^2 - r^2)\alpha^2 > 0;\quad F(q^2) = (q^2 - r^2)\,(q^2 - p^2)\beta^2 < 0;$$
$$F(r^2) = (r^2 - p^2)\,(r^2 - q^2)\gamma^2 > 0.$$

The graph of the function $F(\lambda)$ will therefore have the general character indicated in Fig. 37.* And from it we conclude that the equation $F(\lambda) = 0$ has three real roots, λ_1, λ_2, and λ_3. Hence there are three surfaces of the confocal family which pass through the given point $P(\alpha, \beta, \gamma)$. But we observe also from Fig. 37 that $\lambda_1 < p^2$, $p^2 < \lambda_2 < q^2$, and $q^2 < \lambda_3 < r^2$; therefore, in virtue of Theorem 14, one of these surfaces is an ellipsoid, one an hyperboloid of one sheet, and one an hyperboloid of two sheets.

* We are here assuming that the polynomial $F(\lambda)$ is a continuous function of λ, as in the argument in the footnote on page 290 we assumed that a rational function is continuous except for a finite number of values of the independent variable. A satisfactory proof of these facts is found in treatises on the Theory of Functions of a Real Variable.

If $\alpha = 0$, the root λ_1 becomes equal to p^2, so that in place of the ellipsoid we have the YZ-plane counted doubly; similarly, if $\beta = 0$, the hyperboloid of one sheet is replaced by the double ZX-plane, and if $\gamma = 0$, the hyperboloid of two sheets is replaced by the double XY-plane.

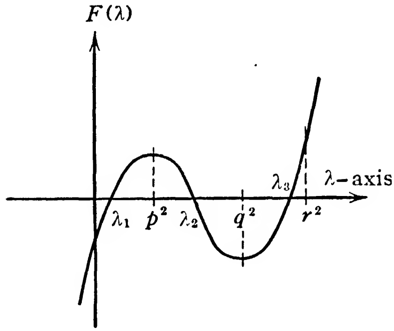

Fig. 37

Corollary. **If $P(\alpha, \beta, \gamma)$ lies on one or more of the coördinate planes, it is still true that three surfaces of every confocal family pass through P; but one or more of the central quadrics of the family are then replaced by the coördinate planes in which P lies.**

Theorem 16. **The three quadrics of a confocal family which pass through an arbitrary point $P(\alpha, \beta, \gamma)$ in space are mutually orthogonal at P.**

Proof. Suppose first that $P(\alpha, \beta, \gamma)$ does not lie on any coördinate plane. Then the three quadrics of the confocal family (1) which pass through P have the equations

$$\frac{x^2}{p^2 - \lambda_i} + \frac{y^2}{q^2 - \lambda_i} + \frac{z^2}{r^2 - \lambda_i} = 1, \quad i = 1, 2, 3,$$

where λ_1, λ_2, and λ_3 are the roots of the equation $F(\lambda) = 0$, discussed above. The equations of the tangent planes to these surfaces at the point P are

$$(3) \quad \frac{\alpha x}{p^2 - \lambda_i} + \frac{\beta y}{q^2 - \lambda_i} + \frac{\gamma z}{r^2 - \lambda_i} = 1, \quad i = 1, 2, 3.$$

Since P lies on each of the three surfaces, we have moreover

$$\frac{\alpha^2}{p^2 - \lambda_i} + \frac{\beta^2}{q^2 - \lambda_i} + \frac{\gamma^2}{r^2 - \lambda_i} = 1, \quad i = 1, 2, 3.$$

If we subtract any two of the last three equations from each other, we find

$$\alpha^2\left[\frac{1}{p^2-\lambda_i}-\frac{1}{p^2-\lambda_j}\right]+\beta^2\left[\frac{1}{q^2-\lambda_i}-\frac{1}{q^2-\lambda_j}\right]$$
$$+\gamma^2\left[\frac{1}{r^2-\lambda_i}-\frac{1}{r^2-\lambda_j}\right]=0, \quad i,j=1,2,3;\ i\neq j.$$

A simple reduction transforms these three equations to the following form:

$$(\lambda_i-\lambda_j)\left[\frac{\alpha^2}{(p^2-\lambda_i)(p^2-\lambda_j)}+\frac{\beta^2}{(q^2-\lambda_i)(q^2-\lambda_j)}+\frac{\gamma^2}{(r^2-\lambda_i)(r^2-\lambda_j)}\right]=0.$$

But since $\lambda_i \neq \lambda_j$, we conclude from this last equation that

$$\frac{\alpha^2}{(p^2-\lambda_i)(p^2-\lambda_j)}+\frac{\beta^2}{(q^2-\lambda_i)(q^2-\lambda_j)}+\frac{\gamma^2}{(r^2-\lambda_i)(r^2-\lambda_j)}=0.$$

And this equation expresses the fact that any two of the tangent planes represented by equations (3) are perpendicular to each other (compare the Corollary of Theorem 9, Chapter IV, Section 46, page 82).

If P lies in a coördinate plane, one of the quadrics of the family which pass through it is that plane itself. Let us suppose that $\alpha = 0$; then the surface in question is the YZ-plane counted doubly. And let the equation

$$\frac{x^2}{p^2-\lambda_2}+\frac{y^2}{q^2-\lambda_2}+\frac{z^2}{r^2-\lambda_2}=1$$

be one of the non-degenerate quadrics passing through P; the tangent plane to this surface at the point $(0, \beta, \gamma)$ is represented by the equation

$$\frac{\beta y}{q^2-\lambda_2}+\frac{\gamma z}{r^2-\lambda_2}=1.$$

This plane is parallel to the X-axis and therefore perpendicular to the double YZ-plane. If P lies on a coördinate axis, two of the quadrics degenerate into double coördinate planes; and these are surely perpendicular. Our theorem has therefore been proved.

* * *

From Theorems 15 and 16, it follows that the quadric surfaces of a confocal family cover the whole of space with a network of

mutually perpendicular surfaces. To each of these surfaces a number is attached, namely, the value of the parameter λ to which it corresponds; and for every point P in space there are three such numbers. These numbers are called the **elliptic space coördinates** of the point P. Our discussion has therefore shown that every ellipsoid $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$ may be made the basis of a system of elliptic space coördinates. We have obtained a frame of reference which generalizes in a remarkable way the Cartesian frames of reference with which we began our study of Solid Analytical Geometry in Chapter III.

And this return to our starting point provides a suitable stopping point, ending in the key in which we began.

In our journey through this book we have examined a few questions in some detail and we have had a glimpse of many things which lay outside our path. It is the author's hope that the reader may have learned to appreciate the beauty and the power of the theory of determinants and matrices, and that he may experience the desire not only to continue the study of the subject to which this book is primarily devoted, but also to enter some of the fields, such as Projective Geometry and the Theory of Functions of a Real Variable, to which we have had occasion to allude now and then in the course of our work.

APPENDIX

I

(*Compare Section* 84, *page* 171)

To prove: **If λ is eliminated from the equations**

$$L_1 = \tfrac{1}{2} \cdot [\lambda Q_1(\alpha, \beta, \gamma) + \mu Q_2(\alpha, \beta, \gamma) + \nu Q_3(\alpha, \beta, \gamma)] = 0$$

and $$L_0 = a_{11}\lambda^2 + a_{22}\mu^2 + a_{33}\nu^2 + 2\, a_{23}\mu\nu + 2\, a_{31}\nu\lambda + 2\, a_{12}\lambda\mu = 0,$$

the resulting quadratic equation in μ and ν is

$$\begin{vmatrix} a_{11} & a_{12} & Q_1 \\ a_{12} & a_{22} & Q_2 \\ Q_1 & Q_2 & 0 \end{vmatrix} \mu^2 + 2 \begin{vmatrix} a_{11} & a_{12} & Q_1 \\ a_{13} & a_{23} & Q_3 \\ Q_1 & Q_2 & 0 \end{vmatrix} \mu\nu + \begin{vmatrix} a_{11} & a_{13} & Q_1 \\ a_{13} & a_{33} & Q_3 \\ Q_1 & Q_3 & 0 \end{vmatrix} \nu^2 = 0.$$

Proof. To simplify the writing we shall treat this problem in a slightly modified form. The given equations are clearly equivalent to the non-homogeneous equations $ax + by + c = 0$ and $p_{11}x^2 + p_{22}y^2 + p_{33} + 2\, p_{23}y + 2\, p_{13}x + 2\, p_{12}xy = 0$ obtained by writing x and y in place of $\frac{\lambda}{\nu}$ and $\frac{\mu}{\nu}$ respectively, and using general coefficients. We assume now that $a \neq 0$ and solve the linear equation for x in terms of y; substitution of the result in the second degree equation leads to the following quadratic in y:

$$p_{11}(by+c)^2 - 2\, p_{12}ay(by+c) + a^2p_{22}y^2 - 2\, p_{13}a(by+c) + 2\, p_{23}a^2y + p_{33}a^2 = 0;$$

upon reduction this equation becomes

$$(p_{11}b^2 - 2\, p_{12}ab + p_{22}a^2)y^2 + 2\, (p_{11}bc - p_{12}ac - p_{13}ab + p_{23}a^2)y + p_{11}c^2 - 2\, p_{13}ac + p_{33}a^2 = 0.$$

Direct expansion of the third order determinants shows that the coefficients differ in sign only from the respective determinants:

$$\begin{vmatrix} p_{11} & p_{12} & a \\ p_{12} & p_{22} & b \\ a & b & 0 \end{vmatrix}, \quad 2 \begin{vmatrix} p_{11} & p_{12} & a \\ p_{13} & p_{23} & c \\ a & b & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} p_{11} & p_{13} & a \\ p_{13} & p_{33} & c \\ a & c & 0 \end{vmatrix}.$$

If we now return to the homogeneous form of the given equations and to the coefficients as given, we have completed the proof.

II

(*Compare Section* 84, *page* 174)

To prove: **If $A_3(Q) = 0$ and $A_{22}(Q) = A_{23}(Q) = A_{33}(Q) = 0$, then every third order minor of the determinant** $\begin{vmatrix} a_{11} & a_{12} & a_{13} & Q_1 \\ a_{12} & a_{22} & a_{23} & Q_2 \\ a_{13} & a_{23} & a_{33} & Q_3 \\ Q_1 & Q_2 & Q_3 & 0 \end{vmatrix}$ **vanishes; $A_3(Q)$ designates the value of this determinant and $A_{ij}(Q)$ the value of the cofactor of the element a_{ij}.**

Proof. We shall use the notation $\overline{Q}_i$ to designate the cofactors of the elements Q_i in the determinant. Since $A_3(Q) = 0$, it follows from the Corollary of Theorem 5, Chapter II (Section 26, page 43) that $A_{11}(Q)A_{22}(Q) - A_{12}^2(Q) = 0$ and $A_{11}(Q)A_{33}(Q) - A_{13}^2(Q) = 0$, so that the hypothesis leads at once to the result that $A_{12}(Q) = A_{13}(Q) = 0$. In the same way we find that $A_{22}A_{44} - \overline{Q}_2^2 = 0$ and $A_{33}A_{44} - \overline{Q}_3^2 = 0$, so that also $\overline{Q}_2 = \overline{Q}_3 = 0$. Moreover $Q_1\overline{Q}_1 + Q_2\overline{Q}_2 + Q_3\overline{Q}_3 = A_3(Q) = 0$; and therefore, since we are working on the hypothesis that $Q_1 \neq 0$ (compare page 171, opening paragraph of Case I), it follows that $\overline{Q}_1 = 0$. Finally we observe that, in virtue of Theorem 13, Chapter I (Section 7, page 13), $Q_1A_{11} + Q_2A_{12} + Q_3A_{13} = 0$ and $a_{11}\overline{Q}_1 + a_{12}\overline{Q}_2 + a_{13}\overline{Q}_3 + Q_1A_{44} = 0$; and from these equations we conclude that $A_{11} = A_{44} = 0$. This completes the proof of our statement.

III

(*Compare Section* 87, *page* 185)

***To prove:* $q(\alpha_{13}, \alpha_{23}, \alpha_{33}) = 0$, if $A_{44} = 0$.**

Proof. Here α_{ij} are the cofactors of the elements a_{ij} in the third order determinant A_{44} and $q(x, y, z)$ is the homogeneous function of the second degree introduced on page 159. By the use of Euler's theorem on homogeneous functions (see footnote on page 161), we find

$$2\,q(\alpha_{13}, \alpha_{23}, \alpha_{33}) = \alpha_{13}q_1(\alpha_{13}, \alpha_{23}, \alpha_{33}) + \alpha_{23}q_2(\alpha_{13}, \alpha_{23}, \alpha_{33}) + \alpha_{33}q_3(\alpha_{13}, \alpha_{23}, \alpha_{33});$$

and

$$\begin{aligned} q_1(\alpha_{13}, \alpha_{23}, \alpha_{33}) &= 2\,(a_{11}\alpha_{13} + a_{12}\alpha_{23} + a_{13}\alpha_{33}) = 0,\\ q_2(\alpha_{13}, \alpha_{23}, \alpha_{33}) &= 2\,(a_{12}\alpha_{13} + a_{22}\alpha_{23} + a_{23}\alpha_{33}) = 0,\\ q_3(\alpha_{13}, \alpha_{23}, \alpha_{33}) &= 2\,(a_{13}\alpha_{13} + a_{23}\alpha_{23} + a_{33}\alpha_{33}) = A_{44} = 0, \end{aligned}$$

by Theorems 13 and 12, Chapter I (Section 7, page 13).

IV

(*Compare Section* 94, *page* 205)

***To prove:* If $r_4 = 1$, then $r_4' < 2$.**

Proof. Here r_4 is the rank of the discriminant matrix of the quadric surface Q and r_4' is the rank of the discriminant matrix of the equation $Q'(x', y', z') = 0$ obtained by rotation of axes from the equation $Q(x, y, z) = 0$.

If $r_4 = 1$, Δ, D_3, and D_2 vanish and therefore, by Theorem 4, Chapter VIII and its Corollary (Section 94, pages 203 and 204), $\Delta' = D_3' = D_2' = 0$. It follows that every three-rowed minor of Δ' vanishes and that the sum of the principal two-rowed minors also vanishes. Since the three-rowed principal minors are themselves symmetric, we can apply to each of them Theorem 7, Chapter II (Section 26, page 44); hence the two-rowed principal minors of any one of the four three-rowed principal minors are of like signs, and since any two of these three-rowed principal minors have a two-rowed principal minor in common, all the principal two-rowed minors have the same sign. It follows then from the fact that $D_2' = 0$ that every two-rowed principal minor of Δ' vanishes. Now we

apply Theorem 6, Chapter II (Section 26, page 43) to each of the three-rowed principal minors; and we conclude that every two-rowed minor of Δ' which is also a minor of a three-rowed principal minor must vanish. It remains to consider the two-rowed minors of Δ' which do not occur in any three-rowed principal minor; the only ones of this kind are the minors

$$\begin{vmatrix} a_{13}' & a_{14}' \\ a_{23}' & a_{24}' \end{vmatrix}, \quad \begin{vmatrix} a_{12}' & a_{14}' \\ a_{32}' & a_{34}' \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_{12}' & a_{13}' \\ a_{42}' & a_{43}' \end{vmatrix},$$

which do not have any element of the principal diagonal of Δ'. To show that these minors vanish also, we consider the three-rowed minors:

$$A_{41}' = \begin{vmatrix} a_{12}' & a_{13}' & a_{14}' \\ a_{22}' & a_{23}' & a_{24}' \\ a_{32}' & a_{33}' & a_{34}' \end{vmatrix} \quad \text{and} \quad A_{31}' = \begin{vmatrix} a_{12}' & a_{13}' & a_{14}' \\ a_{22}' & a_{23}' & a_{24}' \\ a_{42}' & a_{43}' & a_{44}' \end{vmatrix}.$$

These determinants vanish and all their two-rowed minors vanish, except possibly the minors with which we are concerned; and of these, two occur as minors in each of the two three-rowed minors. If we write down the developments of these determinants according to their last rows, we can conclude that the first of these two-rowed minors also vanishes. In a similar way, consideration of the pairs of three-rowed minors A_{21}', A_{41}', and A_{31}', A_{21}' shows that the remaining two-rowed minors vanish. This completes the proof of the proposition.

V

(*Compare Section* 96, *page* 206)

To prove: **The determinant**

$$\begin{vmatrix} 2aa_1 & ab_1 + a_1b & ac_1 + a_1c & ad_1 + a_1d \\ ab_1 + a_1b & 2bb_1 & bc_1 + b_1c & bd_1 + b_1d \\ ac_1 + a_1c & bc_1 + b_1c & 2cc_1 & cd_1 + c_1d \\ ad_1 + a_1d & bd_1 + b_1d & cd_1 + c_1d & 2dd_1 \end{vmatrix}$$

and its three-rowed principal minors vanish.

Proof. Theorem 8, Chapter I (Section 5, page 9) enables us to write this determinant as the sum of 2^4 fourth order determinants whose elements are the product of one of the numbers a, b, c, or d by one of the numbers a_1, b_1, c_1, or d_1. A somewhat careful inspection shows that in every one of these 2^4 determinants at least two columns are proportional; for, after common factors have been removed from the elements of the columns, these columns must consist either of the numbers a, b, c, d, or else of the numbers a_1, b_1, c_1, d_1. Consequently, the value of the given determinant is zero. And every one of the three-rowed principal minors can be written as the sum of 2^3 three-rowed determinants, in each of which there are at least two proportional columns.

The reader should have no difficulty in carrying out the details of this proof; he is urged to write down explicitly a number of the simpler determinants into which those under consideration are broken up.

VI

(*Compare Section* 96, *page* 208)

***To prove:* If $r_4 = 2$, not all the principal two-rowed minors of the matrix a_4 can vanish and those which do not vanish are of one sign.**

Proof. The reader should have no difficulty in proving this statement on the basis of Appendix IV.

VII

(*Compare Section* 105, *page* 241)

***To prove:* The determinant** $\begin{vmatrix} a(p_1^2 - p_2^2) & a(q_1^2 - q_2^2) & \alpha - \alpha_1 \\ b(p_1^2 + p_2^2) & b(q_1^2 + q_2^2) & \beta - \beta_1 \\ 2\,cp_1p_2 & -2\,cq_1q_2 & \gamma - \gamma_1 \end{vmatrix} = 0,$

if for α, β, γ there are substituted the coördinates of an arbitrary point on the line

$$p_1\left(\frac{x}{a} - \frac{y}{b}\right) = p_2\left(1 - \frac{z}{c}\right), \quad p_2\left(\frac{x}{a} + \frac{y}{b}\right) = p_1\left(1 + \frac{z}{c}\right),$$

and for $\alpha_1, \beta_1, \gamma_1$ the coördinates of an arbitrary point on the line

$$q_1\left(\frac{x}{a} - \frac{y}{b}\right) = q_2\left(1 + \frac{z}{c}\right), \quad q_2\left(\frac{x}{a} + \frac{y}{b}\right) = q_1\left(1 - \frac{z}{c}\right).$$

Proof. If we substitute α, β, γ for x, y, z in the equations of the first line, we obtain a pair of linear equations, which may be solved for $\frac{\alpha}{a}$ and $\frac{\gamma}{c}$ by Cramer's rule; for the coefficient determinant of these equations with respect to $\frac{\alpha}{a}$ and $\frac{\gamma}{c}$ is equal to $p_1^2 + p_2^2 \neq 0$. We find

$$\frac{\alpha}{a} = \left[(p_1^2 - p_2^2)\frac{\beta}{b} + 2\,p_1p_2\right] \div (p_1^2 + p_2^2),$$

and

$$\frac{\gamma}{c} = \left[2\,p_1p_2\frac{\beta}{b} + p_2^2 - p_1^2\right] \div (p_1^2 + p_2^2).$$

In a similar manner we obtain from the equations of the second line:

$$\frac{\alpha_1}{a} = \left[(q_1^2 - q_2^2)\frac{\beta_1}{b} + 2\,q_1q_2\right] \div (q_1^2 + q_2^2),$$

and

$$\frac{\gamma_1}{c} = \left[-2\,q_1q_2\frac{\beta_1}{b} + q_1^2 - q_2^2\right] \div (q_1^2 + q_2^2).$$

If we subtract the corresponding equations of these two sets, we can determine $\alpha - \alpha_1$ and $\gamma - \gamma_1$. We substitute these values of $\alpha - \alpha_1$ and $\gamma - \gamma_1$ in the determinant D and make the obvious simplifications; thus we obtain the following result:

$$D = \frac{2\,ac}{(p_1^2 + p_2^2)\,(q_1^2 + q_2^2)} \times$$

$$\begin{vmatrix} p_1^2 - p_2^2 & q_1^2 - q_2^2 & \begin{matrix}(q_1^2 + q_2^2)\,(p_1^2 - p_2^2)\beta - (p_1^2 + p_2^2)\,(q_1^2 - q_2^2) \\ \beta_1 + 2\,bp_1p_2(q_1^2 + q_2^2) - 2\,bq_1q_2(p_1^2 + p_2^2)\end{matrix} \\ p_1^2 + p_2^2 & q_1^2 + q_2^2 & (p_1^2 + p_2^2)\,(q_1^2 + q_2^2)\,(\beta - \beta_1) \\ p_1p_2 & -q_1q_2 & p_1p_2(q_1^2+q_2^2)\beta+q_1q_2(p_1^2+p_2^2)\beta_1+b(p_2^2q_2^2-p_1^2q_1^2) \end{vmatrix}.$$

To the third column of this determinant we add the product of the first column by $-(q_1^2 + q_2^2)\beta$ and the product of the second column by $(p_1^2 + p_2^2)\beta_1$; then we add the second row to the first. Thus we find:

$$D = \frac{4\,abc}{(p_1^2 + p_2^2)\,(q_1^2 + q_2^2)} \times$$
$$\begin{vmatrix} p_1^2 & q_1^2 & p_1p_2(q_1^2 + q_2^2) - q_1q_2(p_1^2 + p_2^2) \\ p_1^2 + p_2^2 & q_1^2 + q_2^2 & 0 \\ p_1p_2 & -q_1q_2 & p_2^2q_2^2 - p_1^2q_1^2 \end{vmatrix}$$
$$= \frac{4\,abc}{(p_1^2 + p_2^2)\,(q_1^2 + q_2^2)} \times$$
$$[(p_2^2q_2^2 - p_1^2q_1^2)\,(p_1^2q_2^2 - p_2^2q_1^2) - p_1^2p_2^2(q_1^2 + q_2^2)^2 + q_1^2q_2^2(p_1^2 + p_2^2)^2]$$
$$= 0.$$

VIII

(Compare Section 109, page 245)

To prove: **If A_1B_1, A_2B_2, . . . are chords of a conic section which pass through a fixed point P and if the products $PA_1 \cdot PB_1$, $PA_2 \cdot PB_2$, . . . are all equal, no matter what point P is taken in the plane of the conic section, then this conic section is a circle.**

Proof. We take a plane Cartesian frame of reference in the plane of the conic section. Let the equation of the conic with respect to this reference frame be

$$C(x, y) = a_{11}x^2 + 2\,a_{12}xy + a_{22}y^2 + 2\,a_{13}x + 2\,a_{23}y + a_{33} = 0,$$

and let $P(\alpha, \beta)$ be an arbitrary point of the plane. We write the equations of an arbitrary line through P in the parametric form as follows:

$$x = \alpha + ls, \quad y = \beta + ms;$$

here s is the parameter which designates the length of the segment of the line from P to the variable point (x, y); $l = \cos\theta$, $m = \sin\theta$, where θ is the inclination of the line. We find then that the distances from P to the points A and B in which the line meets the conic are the roots of the equation

$$s^2c(l, m) + s[C_1(\alpha, \beta)l + C_2(\alpha, \beta)m] + C(\alpha, \beta) = 0,$$

where $c(x, y) = a_{11}x^2 + 2\,a_{12}xy + a_{22}y^2$, and C_1, C_2 are the partial derivatives of $C(x, y)$ with respect to x and y respectively (compare Sections 76 and 80). Therefore $PA \cdot PB = \dfrac{C(\alpha, \beta)}{c(l, m)}$; and we have to show that if $c(l, m)$ is independent of l and m, then the locus of $C(x, y) = 0$ must be a circle.

For $\theta = 0°$, we have $l = 1$, $m = 0$ and $c(l, m) = a_{11}$;
for $\theta = 90°$, we have $l = 0$, $m = 1$ and $c(l, m) = a_{22}$;
for $\theta = 45°$, we have $l = \dfrac{\sqrt{2}}{2}$, $m = \dfrac{\sqrt{2}}{2}$ and $c(l, m) = \dfrac{a_{11}}{2} + a_{12} + \dfrac{a_{22}}{2}$.

Therefore, if $c(l, m)$ is independent of the direction of the line, we must have $a_{11} = a_{22}$, and $a_{12} = 0$. The equation of the conic reduces then to the form

$a_{11}(x^2 + y^2) + 2\,a_{13}x + 2\,a_{23}y + a_{44} = 0$. And if $a_{11} \neq 0$, the locus of this equation is indeed a circle. If $a_{11} = 0$, but a_{13} and a_{23} do not both vanish, the locus is a straight line; and if $a_{11} = a_{13} = a_{23} = 0$, the equation has no finite locus. From the point of view of Projective Geometry, the locus consists, in these two cases of a finite line together with a line at infinite distance, and of a line at infinite distance counted doubly. And these pairs are also recognized as circles; we shall refer to them as degenerate circles.

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